# 39. *Expansion in Bessel Functions.*

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1. Recently J. Delsarte[1] indicated a formal process of expanding arbitrary function, and proposed the question to discuss its convergence. In this paper we will give some sufficient conditions for the convergence of Bessel expansions under his formulation. The method is that used by us in a memoir on Cauchy's series.[2]

Let $f(x)$ be a function defined over $(0, \omega)$ $(\omega > 1)$ and satisfying the following two conditions that (1°) $x^{2p+1}f(x)$ is Lebesque-integrable in $(0, \omega)$, and that (2°) $\lim_{x\to 0} x\, f(x) = 0$.

Let us put, after Delsarte,[3]

$$j(\lambda x) = \frac{2^p \Gamma(p+1)}{(\lambda x)^p} J_p(\lambda x) \tag{1}$$

and

$$L_\lambda\big(f(x)\big) = \frac{\pi}{2x^p} \int_0^x \xi^{p+1}[Y_p(\lambda x) J_p(\lambda \xi) - J_p(\lambda x) Y_p(\lambda \xi)] f(\xi) d\xi . \tag{2}$$

The section of Delsarte series is, then, given by the contour-integral[4]

$$S_r(x;f) \equiv \frac{1}{2\pi i} \oint_{\mathfrak{C}_r} \frac{-2\lambda j_p(\lambda x)}{\delta[j_p(\lambda x)]} \delta\Big[L_\lambda\big(f(x)\big)\Big] d\lambda , \tag{3}$$

where the contour $\mathfrak{C}_r$ is composed of the segment (on the imaginary axis) from $i\rho_r$ to $-i\rho_r$ and the curve $\bar{\mathfrak{C}}_r$ placed in the positive half-plane which starts from $-i\rho_r$ and ends at $i\rho_r$, meeting the real axis at $\tau_r$. Let us designate the parts of $\bar{\mathfrak{C}}_r$ which belong to the first and the fourth quadrants by $\bar{\mathfrak{C}}_r^{(4)}$ and $\bar{\mathfrak{C}}_r^{(1)}$ respectively.

As to the continuity of the linear functionnal $\delta$ we assume the following Property (C): if $\{f_n(x)\}$ is an arbitrary sequence of functions everywhere differentiable in the interval $[0, 1]$, and if the convergences of $\{f_n(x)\}$ and $\{f_n'(x)\}$ to their respective limits $f(x)$ and $f'(x)$ happen at each point of the interval $[0, 1]$, then $\delta[f_n(x)]$ tends to $\delta[f(x)]$.

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1) J. Delsarte: (I) Sur un principe générale de developpement des fonctinos d'une variable réelle en série de fonctions entières. C. R. Paris, **200** (1935). (II) Sur l'application d'un principle général de développement des fonctions d'une variable, aux séries de fonctions de Bessel. C. R. Paris, **200** (1935). (III) Sur un procédé de développement des fonctions en séries et sur quelques applications. J. Math. pures appl., IX. s. **15** 97–102 (1936).

2) T. Kitagawa: On the theory of linear translatable functional equation and Cauchy's series. Japanese Journ. Math. **13** (1937) (under press).

3) See Delsarte (III) pp. 97–100.

4) In Certain points this expression is more general than the Delsarte's formulation where $\delta[j_p(\lambda x)]$ has always simple zero-points. *We assume that $\delta[j_p(\lambda x)]$ does not vanish on the whole imaginary axis including the origin.*

Under these terminologies and the assumptions we will give

Theorem I. *Let there be a sequence of the contour* $\{\mathfrak{C}_r\}$[1] *such that, for any given positive numbers* $\varepsilon$ *and* $\varepsilon'$, *we have, uniformly in the interval* $\varepsilon' \leqq q \leqq 1-\varepsilon$,

(4)

$$\text{(I)} \quad \int_{\tilde{\mathfrak{C}}_r^{(1)}} \left| \frac{\sqrt{\lambda}\, J_p(\lambda q) e^{-iq\lambda}}{\delta\left[\dfrac{J_p(\lambda x)}{x^p}\right]} \right| \left| \delta\left[\frac{H_p^{(1)}(\lambda x)}{x^p}\right] \right| |d\lambda| = O(1)$$

$$\text{(II)} \quad \int_{\tilde{\mathfrak{C}}_r^{(4)}} \left| \frac{\sqrt{\lambda}\, J_p(\lambda q) e^{iq\lambda}}{\delta\left[\dfrac{J_p(\lambda)}{x^p}\right]} \right| \left| \delta\left[\frac{H_p^{(2)}(\lambda x)}{x_p}\right] \right| |d\lambda| = O(1)$$

*and*

$$\text{(III)} \quad \int_{\mathfrak{C}_r} \left| \frac{J_p(\lambda q)}{\delta\left[\dfrac{J_p(\lambda x)}{x^p}\right]} \right| |d\lambda| = O(1),$$

*as r tends to infinity. Then the differences*

$$(5) \qquad S_r(x;f) - \frac{1}{x^p}\int_0^{\tau_r} \lambda J_p(\lambda x)\left(\int_0^{x+d} \xi^{p+1} f(\xi) J_p(\lambda\xi)\, d\xi\right) d\lambda$$

*tend to zero, uniformly in any closed interval interior to the open interval* (0,1), *as r tends to infinity, where we select a positive number d such that* $1+d<\omega$ *and* $d \leqq \varepsilon$.

This theorem contains the well-known convergence-theorems of series of Fourier-Bessel and Bessel-Dini, as one may be convinced if he puts $\delta[f(x)]=f(b)\ (b=1)$ and $\delta[f(x)]=(H+p)f(b)+bf'(b)\ (b=1)$ respectively. In the following we will give the outline of the proof of this theorem.

2. Lemma 1. *Under the hypothesis of Theorem I, the integrals defined by*

$$(6) \quad T_r^{(1)}(x;f) \equiv \frac{1}{2\pi i}\int_{\tilde{\mathfrak{C}}_r^{(1)}} \frac{\lambda J_p(\lambda x)\,\delta\left[\dfrac{H^{(1)}(\lambda x)}{x^p}\right]}{\delta\left[\dfrac{J_p(\lambda x)}{x^p}\right]} \left\{\int_0^{x+d} \xi^{p+1} J_p(\lambda\xi) f(\xi)\, d\xi\right\} d\lambda$$

$$(7) \quad T_r^{(4)}(x;f) \equiv \frac{1}{2\pi i}\int_{\tilde{\mathfrak{C}}_r^{(4)}} \frac{\lambda J_p(\lambda x)\,\delta\left[\dfrac{H^{(2)}(\lambda x)}{x^p}\right]}{\delta\left[\dfrac{J_p(\lambda x)}{x^p}\right]} \left\{\int_0^{x+d} \xi^{p+1} J_p(\lambda\xi) f(\xi)\, d\xi\right\} d\lambda$$

*and*

$$(8) \quad U_r(x;f) \equiv \frac{1}{2\pi i}\int_{\tilde{\mathfrak{C}}_r} \frac{-2\lambda J_p(\lambda x)}{\delta\left[\dfrac{J_p(\lambda x)}{x^p}\right]}\, \delta\left[\frac{\pi}{2x^p}\int_x^{x+d} \xi^{p+1}\{Y_p(\lambda x) J_p(\lambda\xi) - J_p(\lambda x) Y_p(\lambda\xi)\} f(\xi)\, d\xi\right] d\lambda$$

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1) We assume that $\tilde{\mathfrak{C}}_r$ is contained in the region enclosed by $\mathfrak{C}_{r+1}$ and the distances among $\tilde{\mathfrak{C}}_r$ and the origin tend to infinity as $r$ tends to infinity.

*tend to zeros uniformly concerning $x$ in the closed interval $[\eta, 1-\eta']$ as $r$ tends to infinity, $\eta$ and $\eta'$ being any given positive numbers.*

Proof. The asymptotic expansion of the Bessel function[1] $J_p(\lambda x)$ and the Titchmarsh's theorem[2] give us that

$$\int_0^{x+d} \xi^{p+1} J_p(\lambda\xi) f(\xi) d\xi = \frac{1}{\sqrt{\lambda}} \left\{ E_1(\lambda, x) e^{i\lambda(x+d)} + E_2(\lambda, x) \right\} \tag{9}$$

and that, for $\nu = 0, 1$,

$$\frac{d^\nu}{dx^\nu} \left\{ \int_x^{x+d} \xi^{p+1} \left\{ J_p(\lambda\xi) Y_p(\lambda x) - J_p(\lambda x) Y_p(\lambda\xi) \right\} f(\xi) d\xi \right\} = \begin{cases} \dfrac{E_{3,\nu}(\lambda, x)}{\lambda} e^{i\lambda d}, & \text{for } I(\lambda) \leqq 0 \\ \dfrac{E_{4,\nu}(\lambda, x)}{\lambda} e^{-i\lambda d}, & \text{for } I(\lambda) \geqq 0. \end{cases} \tag{10}$$

where $E(\lambda, x)$ denote functions which tend to zero as $|\lambda| \to \infty$.

Consequently, as $r$ tends to infinity, it follows that

$$T_r^{(1)}(x; f) = o\left( \int_{\mathfrak{C}_r^{(1)}} \left| \frac{\sqrt{\lambda}\, J_p(\lambda x) e^{i\lambda x}}{\delta\left[\dfrac{J_p(\lambda x)}{x^p}\right]} \right| \left| \delta\left[\frac{H_p^{(1)}(\lambda x)}{x^p}\right] \right| |d\lambda| \right) \tag{11}$$

$$T_r^{(4)}(x; f) = o\left( \int_{\mathfrak{C}_r^{(4)}} \left| \frac{\sqrt{\lambda}\, J_p(\lambda x) e^{i\lambda(x+d)}}{\delta\left[\dfrac{J_p(\lambda x)}{x^p}\right]} \right| \left| \delta\left[\frac{H_p^{(2)}(\lambda x)}{x^p}\right] \right| |d\lambda| \right) \tag{12}$$

and that, reminding the continuity-property of the linear functionnal $\delta$,

$$U_r(x; f) = o\left( \int_{\mathfrak{C}_r^{(1)}} \left| \frac{J_p(\lambda x) e^{-i\lambda d}}{\delta\left[\dfrac{J_p(\lambda x)}{x^p}\right]} \right| |d\lambda| \right) + o\left( \int_{\mathfrak{C}_r^{(4)}} \left| \frac{J_p(\lambda x) e^{i\lambda d}}{\delta\left[\dfrac{J_p(\lambda x)}{x^p}\right]} \right| |d\lambda| \right). \tag{13}$$

By virtue of the hypothesis (I) (II) and (III), these three relations just obtained lead us to the result, which we were to prove.

Lemma 2. *Under the Hypothesis of Theorem I, we have*

$$\begin{aligned} V_r(x; f) &\equiv \frac{1}{2\pi i} \int_{\mathfrak{C}_r} \frac{-2\lambda j_p(\lambda x)}{\delta[j_p(\lambda x)]} \delta\left[\frac{\pi}{2x^p} Y_p(\lambda x)\right] \left\{ \int_0^{x+d} \xi^{p+1} J_p(\lambda\xi) f(\xi) d\xi \right\} d\lambda \\ &= \frac{1}{x^p} \int_0^{\tau_r} \lambda J_p(\lambda x) \left( \int_0^{x+d} \xi^{p+1} J_p(\lambda\xi) f(\xi) d\xi \right) d\lambda + E_r(x, \lambda; f) \end{aligned} \tag{14}$$

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1) See G. N. Watson: Treatise on the theory of Bessel functions. (1922) chapter VII. pp. 196–200. Specially see 7.21. Asymptotic expansions of $J_\nu(z)$, $J_{-\nu}(z)$ and $Y_\nu(z)$.

2) See E. C. Titchmarsh: Proc. London Math. Soc. **25** (1926).

$$-\frac{e^{-ip\pi}\cos p\pi}{2\pi i}\int_0^{i\rho_r}\frac{2\lambda j_p(\lambda x)}{\delta[j_p(\lambda x)]}\delta\Big[\frac{\pi}{2x^p}J_p(\lambda x)\Big]\Big(\int_0^{x+d}\xi^{p+1}J_p(\lambda\xi)f(\xi)d\xi\Big)d\lambda$$

*where $E_r(x, \lambda; f)$ tends to zero uniformly concerning $x$ in the closed interval $[\eta, 1-\eta']$ as $r$ tends to infinity, $\eta$ and $\eta'$ being any given positive numbers.*

Proof. The decomposition of the integral $V_r(x;f)$ into the two parts corresponding to $\bar{\mathfrak{C}}_r^{(1)}$ and $\bar{\mathfrak{C}}_r^{(4)}$ respectively, gives us, in view of (2),

$$(15)\qquad V_r(x;f)=\frac{1}{2x_p}\int_{\bar{\mathfrak{C}}_r^{(4)}}\lambda J_p(\lambda x)\Big(\int_0^{x+d}\xi^{p+1}J_p(\lambda\xi)f(\xi)d\xi\Big)d\lambda$$
$$-\frac{2\pi i}{x^p}T_r^{(4)}(x;f)$$
$$-\frac{1}{2x^p}\int_{\bar{\mathfrak{C}}_r^{(1)}}\lambda J_p(\lambda x)\Big(\int_0^{x+d}\xi^{p+1}J_p(\lambda\xi)f(\xi)d\xi\Big)d\lambda$$
$$+\frac{2\pi i}{x}T_r^{(1)}(x;f)\,.$$

In the first and the third integrals of the right-hand side let us transform the paths of integration as follows:

$$\int_{\bar{\mathfrak{C}}_r^{(4)}}=\int_{-i\rho_r}^{0}+\int_0^{\tau_r},\qquad \int_{\bar{\mathfrak{C}}_r^{(1)}}=\int_{\tau_r}^{0}+\int_0^{i\rho_r}.$$

Then, reminding the formula $J_p(-\lambda x)=e^{-p\pi i}J_p(\lambda x)$,[1] the sum of these two integrals turns to be equal to the sum of the first two terms of the right-hand side in (14). Hence, in view of Lemma 1, we reach the result to be proved.

Proof of Theorem I. We decompose $\delta\big(\mathfrak{L}_\lambda(f(x))\big)$ into three parts:

$$(16)\quad \delta\big(\mathfrak{L}_\lambda(f(x))\big)=\delta\Big[\frac{\pi}{2x^p}Y_p(\lambda x)\Big]\int_0^{x+d}\xi^{p+1}J_p(\lambda\xi)f(\xi)d\xi$$
$$-\delta\Big[\frac{\pi}{2x^p}J_p(\lambda x)\Big]\int_0^{x+d}\xi^{p+1}Y_p(\lambda\xi)f(\xi)d\xi$$
$$-\delta\Big[\frac{\pi}{2x^p}\int_x^{x+d}\xi^{p+1}\big\{Y_p(\lambda x)J_p(\lambda\xi)-J_p(\lambda x)Y_p(\lambda\xi)\big\}f(\xi)d\xi\Big]$$

This yields us

$$(17)\qquad S_r(x;f)=\frac{1}{2\pi i}\oint_{\mathfrak{C}_r}\frac{-2\lambda j_p(\lambda x)}{\delta\big(j_p(\lambda x)\big)}\delta\Big[\frac{\pi}{2x^p}Y_p(\lambda x)\Big]\Big\{\int_0^{x+d}\xi^{p+1}J_p(\lambda\xi)f(\xi)d\xi\Big\}d\lambda$$

1) See Watson, loc. cit. Chapter III. p. 75. There $z$ is in the first or the second quadrant in the complex $z$-plane.

$$-\frac{1}{2\pi i}\oint_{\mathfrak{C}_r}\frac{-2\lambda j_p(\lambda x)}{\delta\big(j_p(\lambda x)\big)}\delta\left[\frac{\pi}{2x^p}\int_x^{x+d}\xi^{p+1}\Big\{Y_p(\lambda x)J_p(\lambda\xi)-J_p(\lambda x)Y_p)\lambda\xi)\Big\}f(\xi)d\xi\right]d\lambda\,.$$

Remembering the formulae

$$J_p(-z)=e^{-ip\pi}J_p(z)\,;\qquad Y_p(-z)=e^{ip\pi}Y_p(z)-2i\cos p\pi J_p(z)\,,$$

we obtain that the first contour-integral of the right-hand side of (17) is equal to

$$\text{(18)}\qquad \frac{1}{2\pi i}\int_{\bar{\mathfrak{C}}_r}\frac{-2\lambda j_p(\lambda x)}{\delta\big(j_p(\lambda x)\big)}\delta\left[\frac{\pi}{2x^p}Y_p(\lambda x)\right]\left\{\int_0^{x+d}\xi^{p+1}J_p(\lambda\xi)f(\xi)d\xi\right\}d\lambda$$
$$+\frac{e^{-ip\pi}\cos p\pi}{2\pi i}\int_0^{i\rho_r}\frac{2\lambda j_p(\lambda x)}{\delta\big(j_p(\lambda x)\big)}\delta\left[\frac{\pi}{2x^p}Y_p(\lambda x)\right]\left\{\int_0^{x+d}\xi^{J+1}J_p(\lambda\xi)f(\xi)d\xi\right\}d\lambda$$

and that the second contour-integral is equal to

$$\text{(19)}\qquad \frac{1}{2\pi i}\int_{\bar{\mathfrak{C}}_r}\frac{-2\lambda j_p(\lambda x)}{\delta\big(j_p(\lambda x)\big)}\delta\left[\frac{\pi}{2x^p}\int_x^{x+d}\xi^{p+1}\Big\{Y_p(\lambda x)J_p(\lambda\xi)-J_p(\lambda x)Y_p(\lambda\xi)\Big\}f(\xi)d\xi\right]d\lambda\,.$$

Consequently, in virtue of Lemma 2, we reach

$$\text{(20)}\qquad S_r(x\,;f)=\frac{1}{x^p}\int_0^{\tau_r}\lambda J_p(\lambda x)\left(\int_0^{x+d}\xi^{p+1}J_p(\lambda\xi)f(\xi)d\xi\right)d\lambda+E_r(x,\lambda\,;f)$$

which, in view of Hankel's theorem, completes the proof of our theorem.

3. After some modifications of the proof we obtain

Theorem II. *In addition to the assumptions of* $f(x)$ *let us assume that* $f(x)$ *is of bounded variation in* $[0,\omega]$.

*Then in the hypothesis to Theorem I, we may adopt*

(21)

$$\text{(I}'\text{)}\qquad \int_{\mathfrak{C}_r^{(1)}}\left|\frac{J_p(\lambda q)e^{-iq\lambda}}{\sqrt{\lambda}\,\delta\left[\frac{J_v(\lambda x)}{x^p}\right]}\right|\left|\delta\left[\frac{H_p^{(1)}(\lambda x)}{x^p}\right]\right||\,d\lambda\,|=o(1)$$

$$\text{(II}'\text{)}\qquad \int_{\mathfrak{C}_r^{(4)}}\left|\frac{J_p(\lambda q)e^{iq\lambda}}{\sqrt{\lambda}\,\delta\left[\frac{J_p(\lambda x)}{x^p}\right]}\right|\left|\delta\left[\frac{H_p^{(2)}(\lambda x)}{x^p}\right]\right||\,d\lambda\,|=o(1)$$

*and*

$$\text{(III}'\text{)} \qquad \int_{\mathfrak{C}_r} \left| \frac{J_p(\lambda q)}{\lambda \delta \left[ \dfrac{J_p(\lambda x)}{x^p} \right]} \right| |\, d\lambda \,| = o(1)\,,$$

*and the Conclusion remains true.*