# 66. An Extention of the Phragmén-Lindelöf's Theorem.

By Unai MINAMI.

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Theorem 1. *Let $f(z)$ be a function defined in a domain $D$, which satisfies the following conditions:*

1°. *$f(z)$ is holomorphic in $D$.*

2°. *To each point $\zeta$ on the boundary $C$ of $D$ with exception of a point $z_0$, and to each positive number $\varepsilon > 0$, we can associate a circle with the center $\zeta$, in which the following inequality is verified:*

$$|f(z)| \leq m+\varepsilon .$$

3°. *$z_0$ is a limiting point of the boundary $C$ of $D$.*

4°. *In a neighbourhood of $z_0$, $f(z)$ is univalent.*

*Then we have $|f(z)| \leq m$ throughout in $D$.*

Proof. Let us describe a circle $S$ with the center $z_0$; $|z-z_0|=r$ such that $f(z)$ be univalent in the common part of the inside of $S$ and $D$. Then the domain $D$ is decomposed into at most an enumerable infinity of domains, whose boundaries are contained in the boundary $C$ of $D$ and the circle $S$. If the following lemma is established, we can see that in each of those domains, $|f(z)|$ is inferior to a fixed constant (valid for all sub-domains), and therefore, $|f(z)|$ is limited in $D$. Then, applying the Phragmén-Lindelöf's theorem, we can conclude that $|f(z)| \leq m$ throughout in $D$.

Lemma. *Let $f(z)$ be a function defined in $D$ with the following properties:*

1°. *$f(z)$ is holomorphic and univalent in $D$.*

2°. *$z_0$ is a limiting point of the boundary of $D$.*

3°. *For every frontier point $\zeta$ of $D$ distinct from $z_0$, we have*

$$\overline{\lim_{z\to\zeta}}\, |f(z)| \leq m .$$

*Then we have $|f(z)| \leq m$ throughout in $D$.*

Proof of lemma. Let us denote by $\mathfrak{D}$ the set of all the values of $f(z)$, $z$ in $D$. We shall prove first, that there exist a radius $R$ such that we can not trace any Jordan simple closed curve which contains the circle $|w|=R$ inside, and which is situated in $\mathfrak{D}$.

In fact, suppose that there exists no such radius $R$, then we have a sequence of Jordan simple closed curves $C_n (n=1, 2, 3, \ldots\ldots)$, in $\mathfrak{D}$, with the following properties:

1) $C_n$ tend uniformly to $\infty$.

2) $C_{n+1}$ contains $C_n$ inside $(n=1, 2, 3, \ldots\ldots)$.

Then consider the curves $\Gamma_n$ in $D$ such as $C_n$ is image of $\Gamma_n$ by means of $f(z)$. $\Gamma_n$ is any Jordan simple closed curve, and must satisfy the following properties:

1) $\Gamma_n$ tend uniformly to $z_0$.

2) We can choice a subsequence $\Gamma_{n_\nu}$ such that $\Gamma_{n_\nu}$ contains $\Gamma_{n_{\nu+1}}$ in its inside.

Moreover, we can suppose that between $\Gamma_{n_\nu}$ and $\Gamma_{n_{\nu+1}}$ (for all $\nu=1, 2, \ldots\ldots$), there exists at least a frontier point $\zeta_\nu$ of $D$, distinct from $z_0$. Then we can trace a curve in $D$, which starts from $\Gamma_{n_2}$, passes near an accessible frontier point and ends in $\Gamma_{n_3}$, without intersecting any $\Gamma_{n_\nu}$ ($\nu \neq 2, 3$). The image of this curve in $\mathfrak{D}$, goes from $C_{n_2}$ to the point situated near the circle $|w|=m$, and ends in $C_{n_3}$, without intersecting any $C_{n_\nu}$ ($\nu=2, 3$).

This is evidently impossible, because it must intersect $C_{n_1}$. Thus the proposition is demonstrated: there exist a radius $R$, such that we can not trace any Jordan simple closed curve which contains the circle $|w|=R$ inside, and which is situated in $\mathfrak{D}$. Our lemma will be established, if we prove the following theorem:

Theorem 2. *Let $f(z)$ be a function defined in a domain $D$, and denote by $\mathfrak{D}$ the set of all the values of $f(z)$, $z$ in $D$. Suppose that $f(z)$ satisfies the following conditions:*

1°. *$f(z)$ is holomorphic in $D$.*

2°. *To each frontier point $\zeta$, with the exception of a point $z_0$, and to each positive number $\varepsilon > 0$, we can associate a circle with the center $\zeta$, in which the following inequality is verified*

$$|f(z)| \leq m+\varepsilon.$$

3°. *There exist a radius $R$ such that we can not describe any Jordan simple closed curve in $\mathfrak{D}$, which contain the circle $|w|=R$ inside.*

*Then we have $|f(z)| \leq m$ throughout in $D$.*

Proof. Let $R_1$ be any positive number greater than the radius $R$, and describe a circle $|w|=R_1$, with the radius $R_1$. Then we can say that there exists, on this circle, at least one point $w_1$: $|w_1|=R_1$, with the following property: we can not describe any Jordan simple closed curve in $\mathfrak{D}$ which starts from $|w| \leq m+\varepsilon$, contains $w_1$ inside and ends in $|w| \leq m+\varepsilon$, where $\varepsilon$ is any positive fixed number.

In fact, if every point $w$ of $|w|=R_1$ possesses at least one such Jordan simple closed curve in $\mathfrak{D}$, we can describe any Jordan simple closed curve in $\mathfrak{D}$ which contains the circle $|w|=R_1$ inside, which is incompatible with the property of the radius $R < R_1$. Then we have at least two points $w_1$ and $w_2$, $w_1 \neq w_2$, $|w_1|$, $|w_2| > R$, $|w_1|$, $|w_2| > m$, such that we can not describe any Jordan simple closed curve in $\mathfrak{D}$ which contains $w_1$ or $w_2$ inside. Transform $w_1$, $w_2$ and $\infty$ in $w$-plane into $0, 1, \infty$ in $U$-plane by the linear transformation

$$U=l(w)=\frac{w-w_1}{w_2-w_1}.$$

The domain $\mathfrak{D}$ and the circle $|w|=m$ will be transformed into a domain $\Delta$ and a circle $T$ respectively. Let $W=\nu(U)$ be a modular function[1)]

1) For this notation, see p. ex. G. Julia: Lecons sur les fonctions uniformes. Paris, 1924. p. 29.

which transform the domain $U \neq 0, 1$ into $|W| < 1$. Consider the function

$$F(z) = \nu\Big[l\{f(z)\}\Big].$$

As we can not describe any Jordan simple closed curve in $\mathfrak{D}$ which contains $w_1$ or $w_2$ inside, the function $F(z)$ is one-valued[1] and analytic in $D$.

The inside of $T$ in $U$-plane will be transformed into a domain contained in the circle $|W| < \sigma$, $\sigma < 1$. Therefore, for every point $\zeta$ of the frontier $C$ with the exception of $z_0$, the following inequality is verified

$$\overline{\lim_{z \to \zeta}} |F(z)| \leq \sigma < 1$$

$F(z)$ is bounded in $D$. Thus, we have from the Phragmén-Lindelöf's theorem, we have $|F(z)| \leq \sigma$ throughout in $D$, and hence $f(z)$ will be bounded throughout in $D$. The same theorem will show us that $|f(z)| \leq m$ throughout in $D$. Thus, our theorem 2 is proved and consequently the theorem 1 is established.

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1) We can take the some values of the fundamental domain of $\nu^{-1}(W)$ and continuate.