81. On the Exponential-Formula in the Metrical Complete Ring.

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In this note we shall solve the functional equation $\exp(X)\cdot\exp(Y)=\exp\big(Z(X, Y)\big)$ in the *Lie-ring* embedded in the *metrical complete ring*,[1] following after a paper due to F. Hausdorff.[2]

We may replace his *symbolical differentiation* by the differentiation with respect to the *canonical parameters*. The (formal) power series employed in our proof are convergent by the topology defined in the metrical complete ring. In this way the deduction of the final result is much simplified than that of Hausdorff.

The formula $Z(X, Y)$ obtained is expressed in terms of the canonical parameters and the structure-constants of the Lie-ring. It is easy to see that[3] this formula also applies to the ordinary Lie-ring of (analytical) linear differential operators of the first order, for our proof is carried out formally. This constitutes *the converse of the second fundamental theorem of Lie.*

§ 1. Let $\mathfrak{J}$ be a Lie-ring embedded in the metrical complete ring $\mathfrak{R}$. By definition, $\mathfrak{J}$ is a real linear subspace $\leq \mathfrak{R}$ of finite dimension such that we have

$$[X, Y]=XY-YX\in\mathfrak{J} \qquad \text{with } X, Y\in\mathfrak{J}. \tag{1}$$

Consider the functional equation

$$\exp(X)\cdot\exp(Y)=\exp\big(Z((X, Y)\big) \qquad \text{for X, } Y\in\mathfrak{J}, \tag{2}$$

where
$$\exp(A)=\sum_{n=0}^{\infty}(A^n/n!), \qquad A^0=E.$$

It admits a unique solution $Z(X, Y)\in\mathfrak{R}$, if $|X|$, $|Y|$ are sufficiently small, viz.

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1) K. Yosida: On the group embedded in the metrical complete ring. Jap. J. of Math. **13** (1936), p. 7. For the sake of comprehension we will here reproduce the definition of the metrical complete ring.

Let the field of complex numbers be the *Operatorenbereich* of a (non-commutative) ring $\mathfrak{R}$ with the unit E, such that $aA=Aa$ for any $A\in\mathfrak{R}$ and for any complex number a. $\mathfrak{R}$ is called metrical if there is defined a absolute value $|A|$ satisfying the conditions: i) $|A|\geqq 0$, and $|A|=0$ if and only if $A=0$, ii) $|A+B|\leqq|A|+|B|$, $|AB|\leqq|A|\,|B|$ and $|aA|=|a|\,|A|$. The metrical ring $\mathfrak{R}$ is called complete if it is complete in the topology defined by the metric $|A-B|$.

2) F. Hausdorff: Die symbolische Exponentialformel in der Gruppentheorie. Leipziger Berichte, Bd. **58** (1906), p. 19.

3) See Hausdorff: loc. cit.

$$(3) \qquad Z(X, Y)=\ln\left(\exp(X)\cdot\exp(Y)\right)=\sum_{n=1}^{\infty}(-1)^{n-1}\left(\exp(X)\cdot\exp(Y)-E\right)^{n}/n.$$

In reality, we may prove that

$$Z(X, Y)\in\mathfrak{J}.$$

As the infinite series which occur in the following paragraphs are all convergent for sufficiently small $|X|$, $|Y|$, etc. by the topology in $\mathfrak{R}$, the proof will be carried out formally without mentioning of their convergences.

§ 2. *Lemma.*[1] For $X, U\in\mathfrak{R}$ and for a small real number α we have

$$(4) \qquad \begin{cases} \exp(X+\alpha U)=\left(\exp(X)\right)\left(E+\alpha\varphi(U, X)+0(\alpha^2)\right) \\ \qquad\qquad\quad =\left(E+\alpha\psi(U, X)+0(\alpha^2)\right)\left(\exp(X)\right) \end{cases}$$

where

$$(5) \qquad \varphi(U, X)=\psi(U, -X)=U/1!+[U, X]/2!+\big[[U, X], X\big]/3!+\cdots\cdots$$

Proof. Let $F(X)\in\mathfrak{R}$ be any power series in X. We denote by $\overline{F(X+U)}$ the sum of all terms in $F(X+U)$ which have U as one-times factor. Then the first part of (4) is equivalent to

$$\sum_{n=0}^{\infty}\overline{(X+U)^n}/n!=\left(\exp(X)\right)\cdot\varphi(U, X).$$

Thus we have to show

$$(6) \qquad \begin{cases} \dfrac{\overline{(X+U)^n}}{n!}=\dfrac{(UX^{n-1})}{0!\,n!}+\dfrac{X(UX^{n-2})}{1!\,(n-1)!}+\cdots\cdots+\dfrac{X^{n-2}(UX)}{(n-2)!\,2!} \\ \qquad\qquad +\dfrac{X^{n-1}U}{(n-1)!\,1!}, \qquad n=0, 1, 2, \cdots\cdots, \end{cases}$$

where $\qquad (UX)=[U, X], \qquad (UX^2)=\big[[U, X], X\big], \cdots\cdots$

(6) is easily be proved by mathematical induction, remembering the identity

$$\overline{(X+U)^n}/n!=\left(\overline{(X+U)^{n-1}}/(n-1)!\right)\cdot(X/n)+X^{n-1}U/n!.$$

§ 3. Let $V(X, Y)\in\mathfrak{R}$ be defined by the equation

$$\psi(V, Y)=X, \qquad \text{for } X, Y\in\mathfrak{J}.$$

By comparing $\psi(V, Y)=X$ with the numerical equation $v\left(\exp(-y)-1\right)/(-y)=x$, we obtain

1) Hausdorff obtained this Lemma by introducing an unknown symbol W such that $X=[W, X]$.

(7) $$V(X, Y)=c_0X+c_1(XY)+c_2(XY^2)+\cdots\cdots,$$

where the coefficients c_i are given by the ordinary power series

$$y/\big(1-\exp(-y)\big)=c_0+c_1y+c_2y^2+\cdots\cdots$$

Now we have, by (4)

$$\exp\Big(Z\big(X+\alpha X,\ Y-\alpha V(X, Y)\big)\Big)=\big(\exp(X)\big)\ \big(E+0(\alpha^2)\big)\ \big(\exp(Y)\big),$$

and hence

$$\frac{\partial \exp\Big(Z\big(X+\alpha X,\ Y-\alpha V(X, Y)\big)\Big)}{\partial\alpha}=0 \qquad \text{for } \alpha=0.$$

Therefore, by termwise differentiating $\ln\big(\exp(Z)\big)=Z$, we obtain

(8) $$\frac{\partial Z\big(X+\alpha X,\ Y-\alpha V(X, Y)\big)}{\partial\alpha}=0 \qquad \text{for } \alpha=0.$$

§ 4. *We will solve* (8) *with the initial condition* $Z(0, Y)=Y$.

To this purpose, let $X_1, X_2, \ldots\ldots, X_n$ be the linearly independent base of $\mathfrak{F}$. We put $X=\sum_{i=1}^{n} t_iX_i$, $Y=\sum_{i=1}^{n} s_iX_i$. Then by (1), (7)

$$V(X, Y)=\sum_{i=1}^{n} v_i(t_1, t_2, \ldots\ldots, t_n, s_1, s_2, \ldots\ldots, s_n)X_i\in\mathfrak{F},$$

where $v_i(t, s)=v_i(t_1, t_2, \ldots\ldots, t_n, s_1, s_2, \ldots\ldots, s_n)$ is linear homogeneous in $t_1, t_2, \ldots\ldots, t_n$.[1] Thus we have

$$Z\big(X+\alpha X,\ Y-\alpha V(X, Y)\big)=Z\Big((1+\alpha)\sum_{i=1}^{n} t_iX_i,\ \sum_{i=1}^{n}\big(s_i-\alpha v_i(t, s)\big)X_i\Big),$$

and hence by (8)

(9) $$\sum_{i=1}^{n} t_i\frac{\partial Z(t, s)}{\partial t_i}=\sum_{i=1}^{n} v_i(t, s)\frac{\partial Z(t, s)}{\partial s_i}, \qquad Z(t, s)=Z(X, Y).$$

In $Z(t, s)$ let $Z_k(t, s)$ be the term which is homogeneous of k-th degree in $t_1, t_2, \ldots\ldots, t_n$: $Z(t, s)=Z_0\ (t, s)+Z_1(t, s)+\cdots\cdots$

By the initial condition $Z(0, Y)=Y$ we have $Z_0(t, s)=\sum_{i=1}^{n} s_iX_i$. As $v_i(t, s)$ is linear homogeneous in $t_1, t_2, \ldots\ldots, t_n$ we must have

$$kZ_k(t, s)=\sum_{i=1}^{n} v_i(t, s)\frac{\partial Z_{k-1}(t, s)}{\partial s_i},$$

by (9). Hence[2]

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1) See the Remark below.

2) Essentially this formula coincides with that obtained by Hausdorff.

$$(10) \quad Z(X, Y) = Z(t, s) = (E + A/1! + A^2/2! + A^3/3! + \cdots\cdots) \cdot \sum_{i=1}^{n} s_i X_i,$$

where A denotes the differential operator

$$A = \sum_{i=1}^{n} v_i(t, s) \frac{\partial}{\partial s_i}.$$

Thus $Z(X, Y) \in \mathfrak{J}$ with $X, Y \in \mathfrak{J}$, if $|X|$, $|Y|$ are sufficiently small.

Remark. Let the structure-constants of $\mathfrak{J}$ be given by

$$[X_i, X_j] = \sum_{k=1}^{n} c_{ijk} X_k, \qquad (i, j = 1, 2, \ldots\ldots, n).$$

Then, by (5), it is easy to see that we have

$$\begin{Vmatrix} v_1(t, s) \\ v_2(t, s) \\ \vdots \\ v_n(t, s) \end{Vmatrix} = \begin{Vmatrix} \dfrac{\exp\left(-V(s)\right) - E}{-V(s)} \end{Vmatrix}^{-1} \cdot \begin{Vmatrix} t_1 \\ t_2 \\ \vdots \\ t_n \end{Vmatrix},$$

where

$$\begin{cases} V(s) = \| V_{ij}(s) \|, \\ V_{ij}(s) = \sum_{k=1}^{n} s_k c_{jki}, \qquad (i, j = 1, 2, \ldots\ldots, n). \end{cases}$$