105. On the Transformation of a Thetafunction by Siegel's Modular Substitutions.

By Masao SUGAWARA.

(Comm. by T. TAKAGI, M.I.A., Dec. 13, 1937.)

The transformation-formula of a theta-function of the 1st degree by Siegel's modular substitution was already discovered.[1] But in point of view of systematic, it is desirable to deduce it along Siegel's line. Our way is not essentially different from the classical one, but somewhat formally simpler.

Let $M=\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ be a homogeneous modular substitution, that is,

$$(1) \qquad A^*D-C^*B=E, \qquad B^*D=D^*B, \qquad A^*C=C^*A.$$

Then a symmetric matrix Q with positive imaginary part is transformed by the substitution M into a matrix Q_0 of the same kind.

$$Q_0=(AQ+B)\ (CQ+D)^{-1}.$$

We define the thetafunction of the 1st degree with the modulous Q, the argument u, and the characteristics g and h, as follows

$$\vartheta(u;\, g,\, h;\, Q)=\sum_x e^{\pi i(x+g)'Q(x+g)+2\pi i(x+g)'(u+h)},$$

where x runs over the n dimensional lattice points. It is uniformly convergent in u when Q has a positive imaginary part. Now we consider the thetafunction as a function of v, says, $\varphi(v)=\vartheta(u;\, g,\, h;\, Q)$, where $u=(CQ+D)^*v$.

Then the transformation $v \to \begin{cases} v+\underline{E} \\ v+\underline{Q_0} \end{cases}$ corresponds to the transformation $u \to \begin{cases} u+\underline{CQ+D} \\ u+\underline{AQ+B} \end{cases}$ respectively, and

$$\vartheta(u+\underline{CQ+D};\, g,\, h;\, Q)=\sum_x e^{\pi i(x+g+\underline{C})'\,Q(x+g+\underline{C})+2\pi i(x+g+\underline{C})'\,(u+h)}$$
$$\times\ e^{-\pi i\underline{C}'Q\underline{C}-2\pi i\underline{C}'u+2\pi i(g'\underline{D}-h'\underline{C})}$$
$$=\vartheta(u;\, g,\, h;\, Q)\, e^{-\pi i\underline{C}'Q\underline{C}-2\pi i\underline{C}'u+2\pi i(g'\underline{D}-h'\underline{C})},$$

$$\vartheta(u+\underline{AQ+D};\, g,\, h;\, Q)$$
$$=\vartheta(u;\, g,\, h;\, Q)\, e^{-\pi i\underline{A}'Q\underline{A}-2\pi i\underline{A}'u+2\pi i(g'\underline{B}-h'\underline{A})}.$$

Small resp. large letters mean vectors resp. matrices of nth. dimension $\overline{P}$ resp. $\underline{P}$ means νth. column resp. raw-vector of matrix P, where $1 \leqq \nu \leqq n$ is a given number, and $\underline{\overline{P}}$ the νth. diagonal component of P, while $(\underline{\overline{P}})$ represents the vector whose νth. component is $\underline{\overline{P}}$, $(x=1, 2, \ldots\ldots, n)$. $*$ resp.$'$ represents "transposition" of a matrix resp. vector.

1) A. Krazer. Lehrbuch der Thetafunctionen.

Therefore we have

$$\varphi(v+\underline{E})=\varphi(v)\, e^{-\pi i\underline{C}'(\underline{CQ+D}+2u)+2\hat{\underline{g}}\pi i}\,,$$

$$\varphi(v+\underline{Q_0})=\varphi(v)\, e^{-\pi i\underline{A}'(\underline{AQ+B}+2u)-2\hat{\underline{h}}\pi i}\,,$$

where

$$\hat{g}=Dg-Ch+\frac{1}{2}\,(\underline{\overline{CD}^*})\,,$$

$$\hat{h}=-Bg+Ah-\frac{1}{2}\,(\underline{\overline{AB}^*})\,.$$

On the other hand, the function

$$\Psi(v)=e^{\pi iv'C(CQ+D)^*v}$$

has the following properties;

$$\Psi(v+\underline{E})=e^{\pi iv'C(CQ+D)^*v+\pi i\underline{C}'(\underline{CQ+D})+2\pi i\underline{C}'(CQ+D)^*v}$$

$$=\Psi(v)\, e^{\pi i\underline{C}'(\underline{CQ+D}+2u)}\,,$$

$$\Psi(v+\underline{Q_0})=\Psi(v)\, e^{\pi i\underline{Q_0}'C(CQ+D)^*\underline{Q_0}+2\pi i\underline{Q_0}'C(CQ+D)^*v}$$

$$=\Psi(v)\, e^{\pi i(\underline{Q_0}'+2v')C(\underline{AQ+B})}\,,$$

because as M^* is a modular substitution, $CD^*=DC^*$ and thus $C(CQ+D)^*$ is a symmetric matrix.

Therefore the function

$$\Pi(v)=\vartheta(u;\, g,\, h;\, Q)\, e^{\pi iv'C(CQ+D)^*v}$$

has the following properties;

$$\Pi(v+\underline{E})=\Pi(v)\, e^{2\pi i\hat{\underline{g}}}\,,$$

$$\Pi(v+\underline{Q_0})=\Pi(v)\, e^{\pi i(\underline{Q_0}'C-A')(\underline{AQ+B})+2\left((\underline{AQ+B})'C^*-\underline{A}'(CQ+D)^*\right)v-2\hat{\underline{h}}\pi i}$$

$$=\Pi(v)\, e^{-\pi i\underline{\overline{Q_0}}-2\pi i\underline{\bar{v}}-2\pi i\hat{\underline{h}}}\,,$$

because by (1) follow

$$(\underline{AQ+B})'C^*-\underline{A}'(CQ+D)^*=\underline{B}'C^*-\underline{A}'D^*=-\underline{E}'$$

$$(\underline{Q_0}'C-\underline{A}')\,(\underline{AQ+B})=\left((\underline{AQ+B})'C^*-\underline{A}'(CQ+D)^*\right)\underline{Q_0}=-\underline{\overline{Q_0}}\,.$$

$\Pi(v)$ is therefore a thetafunction of the 1st degree with the modulous Q_0 and the characteristic $\hat{g}$ and $\hat{h}$.

As there is only one thetafunction of v of the 1st degree with a given modulous and given characteristics except constant factors, we have

$$\vartheta(u;\, g,\, h;\, Q)\, e^{\pi iv'C(CQ+D)^*v}=K\vartheta(v;\, \hat{g},\, \hat{h};\, Q_0)\,,$$

where K is a constant independent of v.

To determine the constant K we multiply $e^{-2\pi i\hat{g}'v}$ to both sides of the equality

$$\vartheta\big((CQ+D)^*v;\, g,\, h;\, Q\big)\, e^{\pi i v'C(CQ+D)^*v}=K\vartheta(v;\,\hat{g},\,\hat{h};\,Q_0)$$

and integrate with respect x over 0 to 1.

Then by the well-known formula

$$\int_0^1 e^{2x'v\pi i}dv^{1)}=\begin{cases}1, & x=0\\ 0, & x\neq 0,\end{cases}$$

we have

$$\sum_x\int_0^1 e^{\pi i\phi}dv=Ke^{\pi i\hat{g}'Q_0\hat{g}+2\pi i\hat{g}'\hat{h}},$$

where

$$\phi=(x+g)'Q(x+g)+2(x+g)'\big((CQ+D)^*v+h\big)$$
$$+v'C(CQ+D)^*v-2\hat{g}'v.$$

Here we make assumption that the determinant of the matrix C is not zero, that is $|C|\neq 0$, and introduce new summation-vectors y and ρ instead of x by the relation $x=C^*y+\rho$, then the equality

$$\sum_{\rho \bmod. |C|}\sum_y\int_0^1 e^{\pi i\psi}dv=||C||^{n-1}Ke^{\pi i\hat{g}'Q_0\hat{g}+2\pi i\hat{g}'\hat{h}}$$

holds, where

$$\psi=\psi(v;\,y)=(C^*y+\rho+g)'Q(C^*y+\rho+g)$$
$$+2(C^*y+\rho+g)'\big((CQ+D)^*v+h\big)$$
$$+v'C(CQ+D)^*v-2\hat{g}'v$$

and $||C||$ means the absolute value of the determinant $|C|$:

In the summation y runs over whole lattice points, while ρ runs over a complete system of representatives of residues mod. $|C|$.

Now

$$\psi(v;\,y)=\psi(v+y;\,0)-y'CD^*y+2y'Ch+2\hat{g}'y-2(\rho+g)'D^*y$$
$$=\psi(v+y;\,0)-y'CD^*y-2\rho'D^*y+(\underline{CD}^*)'y$$

and

$$y'CD^*y\equiv(\underline{CD}^*)'y \quad \text{mod. } 2.$$

It follows that

$$e^{\pi i\psi(v;\,y)}=e^{\pi i\psi(v+y;\,0)}.$$

Therefore by the elementary formula

---

1) $\int_a^b dv$ means $\int_a^b\int_a^b\cdots\cdots\int_a^b dv_1,dv_2\ldots\ldots dv_n$, where $v=(v_1,v_2,\ldots\ldots,v_n)$.

$$\sum_y \int_0^1 f(y+v)\,dv = \sum_y \int_y^{y+1} f(v)\,dv = \int_{-\infty}^{+\infty} f(v)\,dv\,,$$

we get

$$\sum_{\rho \bmod |C|} \int_{-\infty}^{+\infty} e^{\pi i \psi_0} dv = ||C||^{n-1} K e^{\pi i \hat{g}' Q_0 \hat{g} + 2\pi i \hat{g}' \hat{h}}\,,$$

where

$$\psi_0 = (\rho+g)' Q(\rho+g) + 2(\rho+g)' \big((CQ+D)^* v + h\big)$$
$$+ v' C(CQ+D)^* v - 2\hat{g}' v\,.$$

Put $k = C^{*-1}\big((\rho+g) - (CQ+D)^{-1}\hat{g}\big)$, then by the symmetry of the matrix $C(CQ+D)^*$ we have

$$\psi_0 = (v+k)' C(CQ+D)^* (v+k) - \hat{g}' C^{*-1}(CQ+D)^{-1}\hat{g}$$
$$-(\rho+g)' D^* C^{*-1}(\rho+g) + 2(\rho+g)' h + 2(\rho+g)' C^{-1}\hat{g}\,.$$

If we introduce the expression of $\hat{g}$ in the last term and remember the the relation $D^* C^{*-1} = C^{-1} D$, we get

$$\psi_0 = (v+k)' C(CQ+D)^* (v+k) - \hat{g}' C^{*-1}(CQ+D)^{-1}\hat{g} + g' C^{-1} D g$$
$$-\rho' C^{-1} D \rho + (\rho+g)' C^{-1}(\underline{CD}^*)\,.$$

On the other hand man can easily prove the following formula

$$J = \int_{-\infty}^{+\infty} e^{\pi i (v+k)' C(CQ+D)^* (v+k)} dv = \sqrt{\frac{i^n}{|C(CQ+D)|}}$$ [1]

by bringing the matrix $C(CQ+D)^*$ to its diagonal form.

Hence it follows that

$$\sqrt{\frac{i^n}{|C(CQ+D)|}}\, e^{\pi i (g' C^{-1} D g + g' C^{-1}(\underline{CD}^*))} \sum_{\rho \bmod |C|} e^{-\pi i \rho' C^{-1} D \rho + \pi i \rho' C^{-1}(\underline{CD}^*)}$$
$$= ||C||^{n-1} K e^{\pi i \hat{g}' (Q_0 + C^{*-1}(CQ+D)^{-1})\hat{g} + 2\pi i \hat{g}' \hat{h}}\,.$$

Here

$$Q_0 + C^{*-1}(CQ+D)^{-1} = C^{*-1}(C^* A Q + C^* B + E)\,(CQ+D)^{-1}$$
$$= C^{*-1}(A^* C Q + A^* D)\,(CQ+D)^{-1} = C^{*-1} A^* = A C^{-1}\,.$$

The right hand side of the equality thus becomes

$$||C||^{n-1} K e^{\pi i \hat{g}' A C^{-1} \hat{g} + 2\pi i \hat{g}' \hat{h}}\,.$$

Put $\hat{g}' A C^{-1} \hat{g} + 2\hat{g}' \hat{h} = \varphi + \psi(g, h)$, then

$$\varphi = \frac{1}{4}(\underline{CD}^*)' A C^{-1}(\underline{CD}^*) - \frac{1}{2}(\underline{CD}^*)'\,(\underline{AB}^*)$$

1) C. L. Siegel: Über die analytische Theorie der quadratischen Formen. 1. lemma 37. The sign of the quadratic root may be determined from the primitive formula where $C(CQ+D)^*$ has a diagonal form.

$$\psi(g,h)=(g'D^*-h'C^*)\big((C^{*-1}-B)g+Ah\big)$$
$$-(g'D^*-h'C^*)(\overline{\underline{AB}}{}^*)+(\overline{\underline{CD}}{}^*)'C^{*-1}g\,;$$

because

$$\psi(g,h)=(g'D^*-h'C^*)\big(AC^{-1}(Dg-Ch)-2Bg+2Ah\big)$$
$$-(g'D^*-h'C^*)(\overline{\underline{AB}}{}^*)+(\overline{\underline{CD}}{}^*)\big(AC^{-1}(Dg-Ch)-Bg+Ah\big)$$

and

$$AC^{-1}(Dg-Ch)-Bg+Ah=(C^{*-1}A^*D-B)g$$
$$=\big(C^{*-1}(C^*B+E)-B\big)g=C^{*-1}g\,.$$

The expression of K thus becomes

$$(*)\qquad K=\frac{1}{||C||^{n-1}}\sqrt{\frac{i^n}{|C(CQ+D)|}}\,e^{\pi i\Phi+\Psi(g,h)}$$
$$\sum_{\rho \bmod. |C|} e^{-\pi i\rho'C^{-1}D\rho+\pi i\rho'C^{-1}(\overline{\underline{CD}}{}^*)}\,,$$

where

$$\Phi=-\frac{1}{4}(\overline{\underline{CD}}{}^*)'AC^{-1}(\overline{\underline{CD}}{}^*)+\frac{1}{2}(\overline{\underline{CD}}{}^*)'(\overline{\underline{AB}}{}^*)\,,$$

$$\Psi(g,h)=g'D^*Bg+h'C^*Ah-2h'C^*Bg+(g'D^*-h'C^*)(\overline{\underline{AB}}{}^*)\,.$$

Hence we get at last the

Theorem. Let $M=\begin{pmatrix}A & B\\ C & D\end{pmatrix}$ be a modular substitution, then the following functional equation

$$\vartheta(u;\,g,\,h;\,Q)=Ke^{-\pi i v'C(CQ+D)^*v}\vartheta(v;\,\hat{g},\,\hat{h};\,Q_0)$$

holds, where

$$u=(CQ+D)^*v\,,\qquad Q_0=(AQ+B)(CQ+D)^{-1}\,,$$

$$\hat{g}=Dg-Ch+\frac{1}{2}(\overline{\underline{CD}}{}^*)\,,\qquad \hat{h}=-Bg+Ah-\frac{1}{2}(\overline{\underline{AB}}{}^*)\,,$$

while K has the above expression (*).