10. *Notes on Fourier Series (III): Absolute Summability.*

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1. Let

$$\sum_{n=0}^{\infty} a_n \tag{1}$$

be a series such that

$$\sum_{n=0}^{\infty} a_n \rho^n \tag{2}$$

is convergent for positive $\rho < 1$. We denote (2) by $f(\rho)$. If $f(\rho)$ is of bounded variation in (0,1), that is

$$\int_0^r |f'(\rho)|\, d\rho \qquad (0 < r < 1)$$

is bounded, then we say that (1) is absolutely summable (A) or simply summable $|A|$.[1] The absolutely convergent series is summable $|A|$ and the series summable $|A|$ is summable (A).

Let $f(x)$ be an integrable function, periodic with period 2π, and its Fourier series be

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \equiv \sum_{n=0}^{\infty} A_n(x). \tag{3}$$

Let $\{\lambda_n\}$ be a sequence of real numbers. If the trigonometrical series

$$\sum_{n=0}^{\infty} \lambda_n A_n(x) \tag{4}$$

is summable $|A|$ for almost all x, then $\{\lambda_n\}$ is called the absolutely summable factor of (3).

B. N. Prasad[2] proved that if λ_n is one of the following[3]

$$\frac{1}{(\log n)^{1+\delta}}, \quad \frac{1}{\log n (\log_2 n)^{1+\delta}}, \quad \frac{1}{\log n \log_2 n (\log_3 n)^{1+\delta}}, \ldots\ldots \quad (\delta > 0) \tag{5}$$

then $\{\lambda_n\}$ is the absolutely summable factor. We will prove that if $\{\lambda_n\}$ tends to zero and is convex and further

$$\sum_{n=2}^{\infty} \log n \cdot \Delta\lambda_n \tag{6}$$

converges, then $\{\lambda_n\}$ is an absolutely summable factor. If λ_n tends to zero monotonously, then the convergence of (6) is equivalent to that of

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n}.$$

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1) J. M. Whittaker, Proc. Edinburgh Math. Soc., **2** (1930–1931).
2) B. N. Prasad, Proc. London Math. Soc., **35** (1933).
3) $\log_2 n = \log(\log n)$, $\log_k n = \log(\log_{k-1} n)$ for $k > 2$.

Therefore (5) satisfies the above condition.

In the papers concerning summability $|A|$, Poisson kernel plays the most important rôle. In this paper, applying Abel's transformation some times for the series, we use an elementary property of Fejer's mean only. This makes us to treat the problem easily.

2. *Theorem 1. If $\{\lambda_n\}$ is convex and (6) converges, then $\{\lambda_n\}$ is an absolutely summable factor of Fourier series.*

Let us put

$$g(x,\rho)=\sum_{n=0}^{\infty}\lambda_n A_n(x)\rho^n ,$$

$$J=\int_{\frac{1}{2}}^{r}\left|\frac{\partial}{\partial\rho}g(x,\rho)\right|d\rho , \qquad \frac{1}{2}<r<1 .$$

If we can show that J is bounded as $r\to 1$ for almost all x, then the theorem is proved. We have, by the Abel's transformation,

$$J=\int_{\frac{1}{2}}^{r}\left|\sum_{n=1}^{\infty}n\lambda_n A_n(x)\rho^n\right|d\rho$$

$$=\int_{\frac{1}{2}}^{r}\left|\sum_{n=1}^{\infty}\left\{\sum_{m=1}^{n}mA_m(x)\rho^m\right\}\Delta\lambda_n\right|d\rho ,$$

where $\Delta\lambda_n=\lambda_n-\lambda_{n+1}$. The inner sum is

$$\sum_{m=1}^{n}mA_m(x)\rho^m=\sum_{m=1}^{n-1}\Big(\sum_{\mu=1}^{m}\mu A_\mu(x)\Big)\Delta\rho^m+\rho^n\sum_{\mu=1}^{n}\mu A_\mu(x) .$$

If we denote by $s_n(x)$ and $\sigma_n(x)$ the $(n+1)$-th partial sum and $(n+1)$-th Féjer mean of (3), then

$$\sum_{\mu=1}^{m}\mu A_\mu(x)=(m+1)\{s_m(x)-\sigma_m(x)\}$$

$$=(m+1)t_m(x) , \qquad \text{say.}$$

By the Féjer's theorem the arithmetic mean of $t_n(x)$ is bounded for almost all but fixed x. We have

$$\sum_{m=1}^{n-1}\left\{\sum_{\mu=1}^{m}\mu A_\mu(x)\right\}\Delta\rho^m=\sum_{m=1}^{n-1}(m+1)t_m(x)\Delta\rho^m$$

$$=\sum_{m=1}^{n-2}\left\{\sum_{\mu=1}^{m}(\mu+1)t_\mu(x)\right\}\Delta^2\rho^m+\sum_{\mu=1}^{n-1}(\mu+1)t_\mu(x)\cdot\Delta\rho^{n-1}$$

$$=\sum_{m=1}^{n-2}\left\{(m+1)\sum_{\nu=1}^{m}t_\nu(x)-\sum_{\mu=1}^{m-1}\left[\sum_{\nu=1}^{\mu}t_\nu(x)\right]\right\}\Delta^2\rho^m$$

$$+\left\{n\sum_{\nu=1}^{n-1}t_\nu(x)-\sum_{\mu=1}^{n-2}\left[\sum_{\nu=1}^{\mu}t_\nu(x)\right]\right\}\Delta\rho^{n-1} ,$$

$$\left|\sum_{m=1}^{n-1}\left\{\sum_{\mu=1}^{m}\mu A_\mu(x)\right\}\Delta\rho^m\right|\leqq B_1\left[\sum_{m=1}^{n-2}m^2\Delta^2\rho^m+n^2\Delta\rho^{n-1}\right]$$

$$\leqq B_2\left[\sum_{m=1}^{n-1}\rho^m+n\rho^n+n^2\Delta\rho^n\right]$$

for $\dfrac{1}{2}<\rho<1, B_1, B_2, \ldots\ldots$ being constants independent of n and ρ.

$$\int_{\frac{1}{2}}^{r}\left[\sum_{n=1}^{\infty}\Delta\lambda_n\sum_{m=1}^{n-1}\left\{\sum_{\mu=1}^{m}\mu A_\mu(x)\right\}\Delta\rho^m\Big|\right]d\rho$$

$$\leqq B_3\left[\sum_{n=1}^{\infty}\Delta\lambda_n\sum_{m=1}^{n}\frac{r^m}{m}+\sum_{n=1}^{\infty}r^n\Delta\lambda_n+\sum_{n=1}^{\infty}n\Delta\lambda_n\Delta r^n\right]$$

$$\leqq B_4\left[\sum_{n=1}^{\infty}\log n\cdot\Delta\lambda_n+\lambda_1+\sum_{n=1}^{\infty}\left\{\sum_{m=1}^{n-1}\lambda_m+n\lambda_n\right\}\Delta^2r^n\right]\leqq B_5.$$

$$\int_{\frac{1}{2}}^{r}\left|\sum_{n=1}^{\infty}\rho^n\Delta\lambda_n\sum_{\mu=1}^{n}\mu A_\mu(x)\right|d\rho$$

$$=\int_{\frac{1}{2}}^{r}\left|\sum_{n=1}^{\infty}(n+1)t_n(x)\rho^n\Delta\lambda_n\right|d\rho$$

$$=\int_{\frac{1}{2}}^{r}\left|\sum_{n=1}^{\infty}\left\{\sum_{m=1}^{n}(m+1)t_m(x)\right\}\Delta(\rho^n\Delta\lambda_n)\right|d\rho$$

$$\leqq B_6\int_{\frac{1}{2}}^{r}\left[\sum_{n=1}^{\infty}n^2\Delta(\rho^n\Delta\lambda_n)\right]d\rho$$

$$\leqq B_7\int_{\frac{1}{2}}^{r}\left[\sum_{n=1}^{\infty}n\rho^n\Delta\lambda_n\right]d\rho\leqq B_8\sum_{n=1}^{\infty}r^n\Delta\lambda_n\leqq B_9.$$

Therefore $$J\leqq B_5+B_9.$$

3. *Theorem 2.* *$\{\lambda_n\}$ satisfying the condition in Theorem 1, is the absolutely summable factor of Fourier-Stieltjes series.*

Let the Fourier-Stieltjes series of $g(x)$ be

$$dg(x)\sim\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)$$

and $t_n(x)$ be the analogous one in the proof of Theorem 1. Then

$$\int_0^{2\pi}|\tau_m(x)|\,dx$$

is bounded, $\tau_m(x)$ being the arithmetic mean of $t_n(x)$.

Since J is an increasing function of r, it is sufficient to prove that

$$\int_0^{2\pi}J\,dx$$

is bounded. Hence we can prove Theorem 3 as Theorem 1.

4. We will add a new proof of the following theorem by the former method.

Theorem 3.[1] *For almost all x*

$$\int_0^{r}\left|\sum_{n=1}^{\infty}nA_n(x)\rho^n\right|d\rho=o\left(\log\frac{1}{1-r}\right)\quad\text{as } r\to1.$$

If we use the former notation,

1) B. N. Prasad, loc. cit. and T. Takahashi (=Kawata), Proc. Physic-Math. Soc., **16** (1934).

$$(8)\qquad \sum_{n=1}^{\infty} nA_n(x)\rho^n=\sum_{n=1}^{\infty}\left\{\sum_{\nu=1}^{n}\nu A_\nu(x)\right\}\varDelta\rho^n$$

$$=\sum_{n=1}^{\infty}(n+1)t_n(x)\varDelta\rho^n=\sum_{n=1}^{\infty}\left\{\sum_{\nu=1}^{n}(\nu+1)t_\nu(x)\right\}\varDelta^2\rho^n$$

$$=\sum_{n=1}^{\infty}\left[-\sum_{\nu=1}^{n-1}\left\{\sum_{\mu=1}^{\nu}t_\mu(x)\right\}+(n+1)\sum_{\mu=1}^{n}t_\mu(x)\right]\varDelta^2\rho^n,$$

$$\left|\sum_{n=1}^{\infty} nA_n(x)\rho^n\right|\leqq B_{10}\sum_{n=1}^{\infty}n^2\varDelta^2\rho^n\leqq B_{11}\sum_{n=1}^{\infty}\rho^n,$$

$$\int_0^r\left|\sum_{n=1}^{\infty} nA_n(x)\rho^n\right|d\rho\leqq B_{11}\sum_{n=1}^{\infty}\frac{r^n}{n},$$

that is, the left hand side integral is $O\left(\log\frac{1}{1-r}\right)$. By the elementary way O is replaced by o.