PAPERS COMMUNICATED

24. *An Application of the Fourier Transform to Almost Periodic Function.*

By Shin-ichi TAKAHASHI.
Nagoya College of Technology.
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Prof. B. Jessen has remarked that the theorems of R. Petersen and S. Takahashi on the formal differentiation and integration of the Fourier series of an almost periodic function may be considered as special cases of the following general theorem due to S. Bochner.[1]

Theorem. Let $K(x)$ denote a function of a real variable such that the integral

$$\int_{-\infty}^{\infty} |K(x)|\, dx$$

is convergent, and let $G(\lambda)$ denote its Fourier transform

$$G(\lambda)=\int_{-\infty}^{\infty} K(x)\, e^{-i\lambda x} dx\,.$$

Then if $f(t)$ is an almost periodic function of a real variable with the Fourier series

$$f(t)\sim\sum A_n e^{i\lambda_n t}\,,$$

the series

$$\sum G(\lambda_n)A_n e^{i\lambda_n t}$$

is also the Fourier series of an almost periodic function, namely of the function

$$g(t)=\int_{-\infty}^{\infty} f(-x+t)\, K(x)\, dx\,.$$

By use of this theorem, we can now obtain the following two theorems which correspond to the most general theorems on the formal differentiation and integration of the Fourier series of an almost periodic function.

Theorem 1. Let $f(t)$ be an almost periodic function of a real variable with the Fourier series

$$f(t)\sim\sum A_n e^{i\lambda_n t}\,.$$

Then the two series

$$\sum_{\lambda_n<0} |\lambda_n|^p A_n e^{\lambda_n s}\,, \qquad \sum_{\lambda_n>0} \lambda_n^p A_n e^{\lambda_n s} \qquad (s=\sigma+it)$$

where p is any positive number, are the Dirichlet series of two functions $f_1(s)$, $f_2(s)$ respectively almost periodic in $[0,+\infty)$ and in $(-\infty,0]$.

1) Jessen, Remark on the theorems of R. Petersen and S. Takahashi, Matematisk Tidsskrift B (1935).

Theorem 2. Let $f(t)$ be an almost periodic function of a real variable with the Fourier series

$$f(t) \sim \sum A_n e^{i\lambda_n t}.$$

Then the series

$$\sum \frac{A_n}{(\sigma+i\lambda_n)^p} e^{i\lambda_n t}$$

where σ and p are any positive numbers, is also an almost periodic Fourier series.

Now the theorem 1 corresponds to the choice

$$K(x)=\frac{\Gamma(p)}{2\pi}\cdot\frac{1}{(\sigma+ix)^{p+1}}, \qquad K^*(x)=\frac{\Gamma(p)}{2\pi}\cdot\frac{1}{(\sigma-ix)^{p+1}} \qquad (\sigma>0,\ p>0).$$

Then

$$\int_{-\infty}^{\infty}|K(x)|\,dx=\int_{-\infty}^{\infty}|K^*(x)|\,dx=\pi\Gamma^2\left(\frac{p}{2}\right)\cdot(2\sigma)^{-p}<+\infty$$

and

$$G(\lambda)=\int_{-\infty}^{\infty}K(x)\,e^{-i\lambda x}dx=\begin{cases}0 & \text{for } \lambda>0\\ |\lambda|^p e^{\sigma\lambda} & \text{for } \lambda<0\end{cases}$$

$$G^*(\lambda)=\int_{-\infty}^{\infty}K^*(x)\,e^{-i\lambda x}dx=\begin{cases}\lambda^p e^{-\sigma\lambda} & \text{for } \lambda>0\\ 0 & \text{for } \lambda<0\end{cases}$$

Again the theorem 2 corresponds to the choice

$$K(x)=\begin{cases}\dfrac{1}{\Gamma(p)}x^{p-1}e^{\sigma x} & \text{for } x\geqq 0\\ 0 & \text{for } x<0\end{cases} \qquad (\sigma>0,\ p>0),$$

in which case

$$\int_{-\infty}^{\infty}|K(x)|\,dx=\sigma^{-p}<+\infty$$

and

$$G(\lambda)=\frac{1}{(\sigma+i\lambda)^p}.$$

When the absolute values of the Fourier exponents λ_n have not the point 0 as a limiting point, so that

$$|\lambda_n|\geqq\lambda>0$$

where λ is independent of n, then by Bohr's theorem, the indefinite integrals

$$\int^s f_1(s)\,ds, \qquad \int^s f_2(s)\,ds$$

are almost periodic in $[0, +\infty)$ and $(-\infty, 0]$. Thus we have the following theorem as an immediate corollary of theorem 1.

Theorem 3. Let $f(t)$ be an almost periodic function of a real variable with the Fourier series

$$f(t) \sim \sum A_n e^{i\lambda_n t}$$

where $|\lambda_n|$ have not the point 0 as a limiting point. Then the two series

$$\sum_{\lambda_n<0} \frac{A_n}{|\lambda_n|^p} e^{\lambda_n s}, \qquad \sum_{\lambda_n>0} \frac{A_n}{\lambda_n^p} e^{\lambda_n s} \qquad (s=\sigma+it)$$

where p is any non-negative number, are the Dirichlet series of two functions, almost periodic in $[0, +\infty)$ and $(-\infty, 0]$.