52. *The Characterisations of the Fundamental Operations by Means of the Operational Equations.*

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The problem of determinating the various classes of the fundamental functional operations by means of the operational equations was already discussed by S. Pincherle,[1] but the validity of his results is at least directly limited for the operations whose domains are consisted of analytic functions. We shall extend the results of S. Pincherle to those which are concerned with the linear operations whose domains belong to the function-space $\mathfrak{L}^2(-\infty, \infty)$, abbreviated $\mathfrak{L}^2$, which is consisted of all the real or complex-valued functions which are defined over $(-\infty, \infty)$ and squarely Lebesgue-integrable in any finite range.

The method is that which we used in our previous paper.[2] In stead of normalizing the space $\mathfrak{L}^2$, we shall introduce the family of the section-functions to each element of $\mathfrak{L}^2$, by means of which the notion of the boundedness of the linear operations will be defined. In this sense any multiplication is bounded. This fact makes the application of the functional derivations and the calculus of the generatrix functions easy. In this note we shall communicate our general formulation and the principal results, whose accurate discussion and proofs will be given in another occasion.

§ 1. In what follows we shall denote by E a bounded measurable set of real numbers, with a positive measure, and by $\mathfrak{X}$ the system consisted of all such E. A section-function of $f(x)$ belonging to $\mathfrak{L}^2$, denoted by $f_E(x)$, is meant the one which is defined merely on E and which coincides with $f(x)$ at each point of E, except perhaps for an x-set of zero measure on E. For each assigned E, the function space $[f_E; f \in \mathfrak{L}^2]$[3] constitutes a normalised Banach space with the norm $\|f_E\|_E = \left\{\int_E |f(x)|^2 dx\right\}^{\frac{1}{2}}$, that is to say, $L^2(E)$ space. We shall write $(f, g)_E = \int_E f(x)\overline{g(x)}dx$ and $\|f_E\|_E = \|f\|_E$. Any two functions f_1 and f_2 are said to be equivalent to each other: denoted by $f_1 \cong f_2$ if $f_1(x) = f_2(x)$ in $(-\infty, \infty)$, except perhaps an x-set of measure zero. We mean by a mapping σ of $\mathfrak{X}$ a correspondence of each element E of $\mathfrak{X}$ to another

1) See *S. Pincherle*: Mémoire sur le calcul fonctionnel distributif. Math. Ann. **49** (1897), SS. 325–382. See specially Chapter II, III. Exemples de détermination d'une classe d'operations fonctionnelles au moyen d'une équation symbolique, SS. 356–359.

2) See *T. Kitagawa*: A formulation of the operational calculus on the family of mutually permutable operations. Japanese Journ. Math., **14** (1938), pp. 125–168. See specially Introduction.

3) The set of all elements X with a certain property of $\epsilon(X)$ will be denoted by $[X; \epsilon(X)]$.

uniquely determined element σE with the following two properties: (1°) $E_1 \supset E_2$ implies $\sigma E_1 \supset \sigma E_2$; ($2^\circ$) For any given E from $\mathfrak{X}$ we can find E_1 such that $\sigma E_1 \supset E$.

We shall introduce the fundamental

Definition I. A linear[1] *operation A which transforms each element of* $\mathfrak{L}^2$ *into an element of* $\mathfrak{L}^2$ *is said to be bounded, if the following conditions are fulfilled:*

(1°) *There is a mapping* σ_A *associated with A.*

(2°) *To each given E in* $\mathfrak{X}$, *there corresponds a positive constant* C_E^A *such that the relation holds:*

$$\| Af(x) \|_E \leqq C_E^A \| f(x) \|_{\sigma E}$$

for all f in $\mathfrak{L}^2$, *where* C_E^A *may depend upon A and E, but is independent of f.*

(3°) *To each given E in* $\mathfrak{X}$ *and each given g(x) in* $\mathfrak{L}^2$, *there corresponds a function h(x) in* $L^2(\sigma_A E)$ *such that, for any f in* $\mathfrak{L}^2$, *the relation holds:*

$$(Af, g)_E = (f, h)_{\sigma E},$$

where $h(x) \equiv h_A(x, E, g)$ *may depend upon A, E and g, but is independent of f.*

The connections of our definition with the ordinary boundedness in the L^2-space are worth while to be noted.[2] In the L^2-space there is " one " norm, and any mapping is not needed. Further in that space (3°) is an immediate consequence of (2°) by virtue of the well-known Riesz's theorem, while in our space $\mathfrak{L}^2$ this is not so. Since our notion of boundedness of operations is, it seems, rare in the literatures, we shall give here the concrete examples which illustrate our Definition.

Example 1. The linear operation $M_{\varphi(x)}$ defined by $M_{\varphi(x)} f(x) \cong \varphi(x) f(x)$ with $\varphi(x)$ in $\mathfrak{L}^{2*}$,[3] for each f in $\mathfrak{L}^2$, is bounded. To see this, it suffices to define: (i) $\sigma_{M_\varphi} E = E$; (ii) $C_E^{M_\varphi} = \operatorname*{ess.\,b.}_{x \in E} |\varphi(x)|$; (iii) $h_{M_\varphi}(x, E, g) \cong \overline{\varphi(x)}\, g(x)$ in E.

Example 2. The translation with parameter α, that is, the linear operation T_α defined by $T_\alpha f(x) \cong f(x+\alpha)$ for each f in $\mathfrak{L}^2$, is bounded. To see this, it suffices to define: putting $\tau_\beta E = [x+\beta;\, x \in E]$, that is to say, the set of all $x+\beta$ as x ranges through E, $\sigma_{T_\alpha} E$ is defined $\sigma_{T_\alpha} E = \sum_{-\alpha \leq \beta \leq \alpha} \tau_\beta E$; (ii) $C_E^{T_\alpha} = 1$; (iii) $h_{T_\alpha}(x, E, g) = g(x-\alpha)$ in $\tau_\alpha E$ and it vanishes at each point of $\sigma_{T_\alpha} E - \tau_\alpha E$.

A linear operation which is not bounded in our sense is said to be unbounded. Each of the following two operations is unbounded: (α) $Af(x) = f'(x)$; (β) $Af(x) = \int_{-\infty}^{\infty} f(x+t)\, dt$.

1) A is said to be linear, if, whenever, $f_1(x)$, $f_2(x)$ belong to the domain of A, $\mathfrak{D}(A)$, then for any pair of two real or complex numbers α and β, $\alpha f_1(x) + \beta f_2(x)$ belongs to $\mathfrak{D}(A)$, and $A(\alpha_1 f + \beta f_2) = \alpha A f_1 + \beta A f_2$.

2) In my previous paper loc. cit., I did not assume the property (3°). But it has been found that the employment of the property (3°) makes our arguments much easy.

3) $\mathfrak{L}^{2*} = [f;\, f \in \mathfrak{L}^2$ and are essentially bounded in any finite range$]$.

§ 2. For any two linear operations A, B the commutator-product of A and B, denoted by $[A, B]$, is defined by $[A, B]f = ABf - BAf$ in case when the right members have their meaning at all. After S. Pincherle[1] the first functional derivation of a linear operation A, defined by $A^{(1)} = [A, M]$, where M is an abbreviated notation of the multiplication M_x. The functional derivations of the higher order are defined by inductory way; if we have defined $A^{(n-1)}$ with $n \geqq 2$, then $A^{(n)}$ will be defined by $A^{(n)} = [A^{(n-1)}, M]$.

Under an operational equation we shall mean an equation of the form:

$$\varphi\big(x, B_1 f(x), B_2 f(x), \dots, B_m f(x), Af, A^{(1)}f, \dots, A^{(n)}f\big) = 0\,,$$

where the functional relation φ and the operations B are known and the equality holds in $-\infty < x < \infty$, except perhaps an x-set of measure zero, for any function f for which the left-hand side member has the meaning at all. In the following lines we may and we shall omit f in the above equation and write simply:

$$\varphi(x, B_1, B_2, \dots, B_m, A^{(1)}, A^{(2)}, \dots, A^{(n)}) = 0\,.$$

§ 3. After this preparation we are now in a position to communicate the following theorems.

Theorem I.[2] *A bounded linear operation A satisfies the operational equation $A^{(1)} = 0$, if and only if there is a function $\varphi(x)$ in $\mathfrak{L}^{2*}$ such that $A \equiv M_{\varphi(x)}$.*

Theorem II. A bounded linear operation A satisfies the operational equation $A^{(1)} = \alpha A$ with an assigned real number α, if and only if $A \equiv M_{\varphi(x)} T_\alpha$ with $\varphi(x)$ in $\mathfrak{L}^{2*}$, that is, $Af(x) \cong \varphi(x) f(x+\alpha)$ for each $f(x)$ in $\mathfrak{L}^2$.

A linear operation A is said to be *translatable* in $\mathfrak{D}(A)$ contained in $\mathfrak{L}^2$, if, for each real number α, $f \in \mathfrak{D}(A)$ implies that $T_\alpha f \in \mathfrak{D}(A)$ and $T_\alpha A f \cong A T_\alpha f$. In case when $e^{\lambda x}$ belongs to $\mathfrak{D}(A)$ for each λ belonging to a set Λ, $g_A(\lambda, x)$, which is defined by $Ae^{\lambda x} \cong g_A(\lambda, x) e^{\lambda x}$, denotes a function belonging to $\mathfrak{L}^2$ for each λ in Λ. As a direct consequence of this definition we see that to a translatable operation A there corresponds a function $g_A(\lambda)$ such that $g_A(\lambda, x) = g_A(\lambda)$ in $-\infty < x < \infty$, except perhaps an x-set of measure zero (depending on λ in general), for each λ for which $e^{\lambda x} \in \mathfrak{D}(A)$.

Theorem III. For a bounded linear operation A, the following two conditions are equivalent to each other; that is, each one of them implies the other:

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1) See Pincherle loc. cit., Chapter II, II. Derivation fonctionnelle. Dévelopement fonctionnel de Taylor, SS. 352–359.

2) Cf. *F. J. Murray and J. v. Neumann*: On rings of operators. Annals Math., **37** (1936), pp. 116–229. Theorem I is to be compared with Lemma 12. 2. 2 in pp. 196–197. It is to be noted that, in our formulation, M_x is a bounded operation and each x^n belong to $\mathfrak{L}^2$, while in the Hilbert space $L^2(-\infty, \infty)$ M_x is unbounded and none of x^n belongs to the space when $n \geqq 0$.

Further it is to be noted that our Theorem I is a special case of Theorem II. It is rather for the sake of emphasizing the contrast between multiplications and linear translatable operations.

(i) *A is translatable.*

(ii) *For each complex number λ, $g_A(\lambda, x)=g_A(\lambda)$ in $-\infty<x<\infty$, except perhaps an x-set of measure zero, and $g_A(\lambda)$ is an integral function of λ.*[1)]

We can proceed to observe

Theorem IV. A linear operation A which is defined for every $f(x)$ in $\mathfrak{L}^2$ is bounded and translatable, if and only if it satisfies the following three conditions:

(1°) *For each λ in the complex λ-plane, $g_A(\lambda, x)=g_A(\lambda)$ in $-\infty<x<\infty$, except perhaps an x-set of measure zero (depending on λ in general), and $g_A(\lambda)$ is an integral function of λ.*

(2°) *There is a positive number N_A, independent of ν, such that $|g_A(i\nu)|\leqq N_A$ in $-\infty<\nu<\infty$.*

(3°) *To each given E, there corresponds an interval (a, b) which contains E and such that for any f in $\mathfrak{L}^2$, we have*

$$\lim_{n\to\infty}\left\| \sum_{k=-n}^{n} \frac{e^{i\lambda_k x}}{(b-a)e^{i\lambda_k a}} R_A^{i\lambda_k} f(b)\right\|_E = 0,$$

where we put

$$R_A^\lambda f(x) = e^{\lambda x}\int_a^x e^{-\lambda\eta} Af(\eta)\,d\eta - g_A(\lambda)e^{\lambda x}\int_a^x e^{-\lambda\eta}f(\eta)\,d\eta^{2)}$$

$$\lambda_k = 2k\pi/(b-a). \qquad (k=0, \pm 1, \pm 2, \dots)$$

Further if $E_1\supset E_2$, the corresponding intervals (a_1, b_2) and (a_2, b_2) are subject to $(a_1, b_1)\supset(a_2, b_2)$.

Corollary. With a bounded linear translatable operation A we can associate a mapping σ_A such that, for each f in $\mathfrak{L}^2$, $\| Af(x)\|_E \leqq N_A \|f\|_{\sigma_A E}$, where $N_A = \underset{-\infty<\nu<\infty}{\text{l. u. b.}} |g_A(i\nu)|$.

§ 4. For a bounded linear operation A, $A_{(a_1, a_2, \dots, a_n)}$ $(n=1, 2, 3, \dots)$ are defined by induction as follows: (i) for each real number a, $[T_a, A]\equiv A_{(a)}$; (ii) when $A_{(a_1, a_2, \dots, a_{n-1})}$ has been already defined for any set of $n-1$ real numbers $(a_1, a_2, \dots, a_{n-1})$, then $A_{(a_1, a_2, \dots, a_n)}$ is defined by $A_{(a_1, a_2, \dots, a_n)}\equiv[T_{a_n}, A_{(a_1, a_2, \dots, a_{n-1})}]$ for any real number a_n. We observe

Theorem V. A bounded linear operation A satisfies the equation $A_{(a_1, a_2, \dots, a_n)} = n!\ a_1 a_2 \cdots a_n\ T_{a_1+a_2+\dots+a_n}$ for any set of n real numbers $(a_1, a_2, \dots, a_n)$, if and only if there is a system of the n bounded linear translatable operations A_k $(k=0, 1, 2, \dots, n-1)$ such that $A\equiv M^n + A_{n-1}M^{n-1} + A_{n-2}M^{n-2} + \cdots A_0 M$.

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1) It is by no means true that to any assigned integral function $g(\lambda)$ there corresponds a *bounded* linear translatable operation A such that $Ae^{\lambda x} \cong g(x)e^{\lambda x}$. Cf., the following Theorem IV.

2) The rôle of the operation R_A^λ may be seen from the discussions developed in my previous paper loc. cit., pp. 158–160, where the notation H_B^λ is adopted in stead of R_A^λ.

A bounded linear operation A characterised by Theorem V may be called to be *of the Laplace type*, in connection with the terminology adopted in the theories of Laplace differential equation of infinite order.[1] It forms a contrast with differential operation, as a bounded linear translatable operation with a multiplication. The differentiation, however, cannot be bounded: indeed we can prove that a linear operation A which satisfies the operational equation $A^{(1)}=1$ cannot be bounded. Therefore it is necessary to introduce

Definition II. $\mathfrak{B}$ *is the linear set of all the* $f(x)$ *in* $\mathfrak{L}^2$ *for which there is a function* $f^*(x)$ *in* $\mathfrak{L}^2$ *such that, for each* E, $\lim_{a\to 0} \| (T_a f(x)-f(x))/a-f^*(x) \|_E=0$. *The linear operation* D *is defined by*: $Df(x) \cong f^*(x)$, *for* $f(x)$ *in* $\mathfrak{B}$. $\mathfrak{B}^{(n)}$ *is the linear space of those functions* $f(x)$ *in* $\mathfrak{L}^2$ *for which the* $(n+1)$*-equivalence relations*: (i) $f_{(0)}(x) \cong f(x)$; (ii) $f_{(1)}(x) \cong Df_{(0)}(x)$; (iii) $f_{(2)}(x) \cong Df_{(1)}(x)$; $\dots$; $(n+1)$ $f_{(n)}(x) \cong Df_{(n-1)}(x)$, *define* $f_{(r)}(x)$ *which are absolutely continuous for* $0 \leqq r < n$ *and belongs to* $\mathfrak{L}^2$ *for* n. *We define the linear operation* D^n *by*: $D^n f(x) \cong f_{(n)}(x)$ *for* $f(x)$ *in* $\mathfrak{B}^{(n)}$.

Definition III. A linear operation A *is said to be bounded in the depth* n *concerning the mapping* σ_A *if the following conditions are fulfilled*:

(1°) $\mathfrak{D}(A)=\mathfrak{B}^{(n)}$ *and there corresponds a mapping* σ_A *to* A.

(2°) *To each given* E, *there corresponds* $\sigma_A E$ *such that, for any* f *in* $\mathfrak{B}^{(n)}$,

$$\| Af(x) \|_E \leqq C^A_{0,E} \| f(x) \|_{\sigma_A E}+C^A_{1,E} \| f(x) \|_{\sigma_A E}+\cdots C^A_{n,E} \| f(x) \|_{\sigma_A E},$$

where the non-negative numbers $C^A_{k,E}(k=0, 1, 2, \dots, n)$ *may depend upon* E *and* A, *but are independent of* E.

(3°) *To each given* $g(x)$ *in* E *and each given* $g(x)$ *in* $\mathfrak{B}^{(n)}$, *we can find a function* $h(x)$ *such that*

$$(Af, g)_E=(f, h)_{\sigma_A E}$$

where $h(x)\equiv h_A(x, E, g)$ *may depend upon* A, E *and* g, *but is independent of* f *in* $\mathfrak{B}^{(n)}$.

Then we observe

Theorem VI. A linear operation A *which is bounded in the depth* n *satisfies the equation* $A^{(n)}=n!$ *if and only if it is a linear differential operation*[2] *of the exactly* n*-th order; that is, for each* $f(x)$ *in* $\mathfrak{B}^{(n)}$, $Af(x) \cong g_{(n)}(x)$, *where* $g_{(n)}(x)$ *is defined by the following system of equivalences*; (i) $g_{(0)}(x) \cong \varphi_0(x) f(x)$; (ii) $g_{(k)}(x) \cong g_{(k-1)}(x)+\varphi_k(x) D^k f(x)$ $(k=1, 2, \dots, n)$, *where* $\varphi_0(x), \varphi_1(x), \dots, \varphi_n(x)$ $(\equiv 1)$ *are all belonging to* $\mathfrak{L}^{2*}$.

In a similar manner we can give a general form of the linear operations which are bounded in the depth n and which satisfy the

1) See *H. T. Davis*: The theory of linear operations from the standpoint of differential equation of infinite order. The Principia Press (1936). Specially see p. 338.

2) See *M. Bôcher*: Leçons sur les Méthodes de Sturm. Gauthier-Villars et cie, Paris (1917). Also cf. *Israel Halperin*: Closures and adjoints of linear differential operators. Annals Math. **38** (1937).

equation $A^{(n)}+\rho_{n-1}(x)A^{(n-1)}+\cdots+\rho_0(x)A=0$, where $\rho_k(k)$ are given functions belonging to $\mathfrak{L}^{2*}$.

§ 5. The chief aim of our investigations lies in establishing a formulation which is suitable for the researches of the associated functional equations. We shall apply the results of this paper on our research on the linear functional equations of the Laplace type, which the author is now preparing to issue.[1)]

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1) In a characterisation of Schrödinger's operator in quantum mechanics, J. v. Neumann appealed to the relation due to H. Weyl concerning the two classes of unitary operators $U(\alpha)=\exp(2\pi i\alpha P/h)$ and $V(\beta)=\exp(2\pi i\beta Q/h)$, in stead of the employment of the classical relation $PQ-QP=h1/2\pi i$. Our present formulation may be recognised to be an intermediate one between the two formulations.

See *J. v. Neumann*: Die Eindeutigkeit der Schrödingeren Operatoren. Math., Ann. **104** (1929), SS. 570–578.