PAPERS COMMUNICATED

51. *A Theorem Concerning the Non-vanishing of Functions.*

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1. As a typical result concerning the non-vanishing of functions, Mr. N. Levinson[1] has proved the following theorem:

If $f(x)$ is the Fourier transform of a function $G(u)\in L_1(-\infty,\infty)$, and for large u, $G(u)=O(e^{-\theta(u)})$, where $\theta(u)$ is non-decreasing function for $u>0$ such that

$$\int_1^\infty \frac{\theta(u)}{u^2}\,du=\infty\,,$$

then $f(x)$ can not be zero in any interval unless it is identically zero in $(-\infty,\infty)$.

We will in the present paper prove a theorem of similar type not assuming the condition concerning the rapidity of tending to zero for $G(u)$, but supposing that $G(u)$ vanishes except in a sequence of intervals for $u>A$.

If $G(u)$ vanishes for positive u, then $f(x)$ is the boundary function of a function analytic in the upper half-plane which can be represented as Cauchy or Poisson integral. This is the important result due to Professors Hille and Tamarkin.[2] From this theorem it results immediately that $f(x)$ can not vanish in a certain interval unless it vanishes identically. The theorem we prove is the generalization of this fact to the case where $G(u)$ does not necessarily vanish for all $u>0$.

Our theorem can be also considered as the Fourier transform analogue of a gap theorem for Fourier series due to Paley and Wiener.[3] We follow the line of argument used by them.

2. Our theorem runs as follows.

Theorem. *Suppose that $f(x)$ is the Fourier transform of a func-*

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1) N. Levinson, On a class of non-vanishing functions, Proc. London Math. Soc., (2) **41** (1936).

2) E. Hille and J. D. Tamarkin, On a theorem of Paley and Wiener, Annals of Math., (2) **34** (1933).

" " " , A remark on Fourier transforms and functions analytic in a half-plane, Compositio Math., **1** (1934).

" " " , On the absolute integrability of Fourier transforms, Fund. Math., **25** (1935).

3) Paley and Wiener, Fourier transforms in the complex domain, Amer. Colloq. XIX, p. 123. Theorem 42.

N. Wiener, A class of gap theorems, Annali di Pisa, (2) **3** (1934).

See also T. Kawata, A gap theorem for the Fourier series of an almost periodic function, Tôhoku Math. Journ., **43** (1937).

tion $G(u)$ which belongs to $L_1(-\infty,\infty)$ and $L_2(-\infty,\infty)$ and $G(u)$ vanishes for $u>0$ except in a sequence of non-overlapping intervals $I_n=(\mu_n-1,\ \mu_n+1)$ $(n=1, 2, \ldots\ldots)$, where

$$\lim_{n\to\infty}(\mu_{n+1}-\mu_n)=\infty\,, \tag{1}$$

and further that

$$0<A<\frac{G(u)}{G(v)}<B\,, \tag{2}$$

for $\mu_n-1<u,\ v<\mu_n+1$, where A and B are constants independent of n. Then $f(x)$ can not vanish in any interval unless it vanishes identically in $(-\infty,\infty)$.

If besides the conditions of the theorem, $G(u)$ is zero for $-\infty<u<\infty$ except in the non-overlapping intervals $(\mu_n-1,\ \mu_n+1)$ $(n=\pm1, \pm2, \ldots\ldots)$, then we say that $f(x)$ satifies the condition $\mathfrak{A}$.

We can prove this theorem easily from following two lemmas.

Lemma 1. If $f(x)$ satisfies the condition $\mathfrak{A}$, and $|\mu_n-\mu_{n+1}|>L>4$, then we have

$$\int_{-\frac{C_1}{L}}^{\frac{C_1}{L}}|f(x)|^2dx\geqq C_2\int_{-\frac{\pi}{L}+y}^{\frac{\pi}{L}+y}|f(x)|^2dx \tag{3}$$

for every y, where $C_1=8A+\pi$, $C_2=1/A(8A+\pi)$.

$$\text{Since}\quad \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{iux}e^{-i\lambda x}\frac{\sin\frac{Lx}{2}}{x}dx=\begin{cases}\sqrt{\frac{\pi}{2}}\,, & \text{for}\quad |\lambda-u|<\frac{L}{2}\,,\\ 0\;, & \text{for}\quad |\lambda-u|>\frac{L}{2}\,,\end{cases}$$

it results that

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{iux}G(\mu_n+t)\,e^{-i(\mu_n+t)}\frac{\sin\frac{Lx}{2}}{x}dx \tag{4}$$

$$=\begin{cases}\sqrt{\frac{\pi}{2}}G(\mu_n+t)\,, & \text{for}\quad |u-(\mu_n+t)|<\frac{L}{2}\,,\\ 0\qquad , & \text{for}\quad |u-(\mu_n+t)|>\frac{L}{2}\,.\end{cases}$$

Application of Plancherel's theorem shows that

$$\int_{-\infty}^{\infty}|\sum_{n=-N}^{N}G(\mu_n+t)\,e^{-i(\mu_n+t)x}|^2\frac{\sin^2\frac{Lx}{2}}{x^2}dx$$

$$=\pi L\sum_{n=-N}^{N}G^2(\mu_n+t)\,.$$

Integrating both sides with respect to t and applying the Schwarz's inequality, we obtain

$$\frac{\pi L}{2}\int_{-1}^{1}\sum_{-N}^{N} G^2(\mu_n+t)\,dt$$

$$=\int_{-\infty}^{\infty}\frac{\sin^2\frac{Lx}{2}}{x^2}\,dx\int_{-1}^{1}\Big|\sum_{-N}^{N} G(\mu_n+t)\,e^{-i(\mu_n+t)}\Big|^2 dt$$

$$\geqq\frac{1}{2}\int_{-\infty}^{\infty}\frac{\sin^2\frac{Lx}{2}}{x^2}\,dx\left|\int_{-1}^{1}\sum_{-N}^{N} G(\mu_n+t)\,e^{-i(\mu_n+t)x}\,dt\right|^2.$$

That is

$$\frac{\pi L}{2}\sum_{-N}^{N}\int_{\mu_n-1}^{\mu_n+1} G^2(u)\,du\geqq\frac{1}{2}\int_{-\infty}^{\infty}\frac{\sin^2\frac{Lx}{2}}{x^2}\left|\sum_{-N}^{N}\int_{\mu_n-1}^{\mu_n+1} G(u)\,e^{-iux}\,du\right|^2 dx.$$

Letting $N\to\infty$ we have

$$\int_{-\infty}^{\infty} G^2(u)\,du\geqq\pi^{-1}L^{-1}\int_{-\infty}^{\infty}\frac{\sin^2\frac{Lx}{2}}{x^2}\left|\int_{-\infty}^{\infty} G(u)\,e^{-iux}\,du\right|^2 dx,$$

that is:

$$\int_{-\infty}^{\infty} G^2(u)\,du\geqq\pi^{-1}L^{-1}\int_{-\infty}^{\infty}|f(x)|^2\frac{\sin^2\frac{Lx}{2}}{x^2}\,dx. \tag{5}$$

Integrating both sides of (4) with respect to t from -1 to 1, we get

$$(2\pi)^{-\frac{1}{2}}\int_{-\infty}^{\infty} e^{iux}\frac{\sin\frac{Lx}{2}}{x}\,dx\int_{-1}^{1} G(\mu_n+t)\,e^{-i(\mu_n+t)x}\,dt$$

$$=\begin{cases}\sqrt{\frac{\pi}{2}}\displaystyle\int_{-1}^{1} G(\mu_n+t)\,dt, & \text{for } |u-\mu_n|<\frac{L}{2}-1,\\[2ex] \sqrt{\frac{\pi}{2}}\displaystyle\int_{1}^{u-\mu_n+\frac{L}{2}} G(\mu_n+t)\,dt, & \text{for } \mu_n-\frac{L}{2}-1<u<\mu_n-\frac{L}{2}+1,\\[2ex] \sqrt{\frac{\pi}{2}}\displaystyle\int_{u-\mu_n-\frac{L}{2}}^{1} G(\mu_n+t)\,dt, & \text{for } \mu_n+\frac{L}{2}-1<u<\mu_n+\frac{L}{2}+1,\\[2ex] 0, & \text{otherwise,}\end{cases}$$

$$\equiv\varphi(u),\quad\text{say.}$$

$\varphi(u)$ vanishes except in $\mu_n-\frac{L}{2}-1<u<\mu_n+\frac{L}{2}+1$ $(n=0,\pm1,\pm2,\ldots)$

and by the condition (2) $G(u)$ takes the same sign in every interval $\mu_n-\frac{L}{2}<u<\mu_n+\frac{L}{2}$. From Plancherel's theorem we have

$$(6)\qquad \int_{-\infty}^{\infty}\frac{\sin^2\frac{Lx}{2}}{x^2}dx\left|\sum_{-N}^{N}\int_{-1}^{1}G(\mu_n+t)e^{-i(\mu_n+t)x}\right|^2$$

$$=\int_{-\infty}^{\infty}\left|\sum_{-N}^{N}\varphi_n(u)\right|^2du$$

$$\geqq\frac{\pi}{2}(L-2)\sum_{-N}^{N}\left|\int_{-1}^{1}G(\mu_n+t)dt\right|^2.$$

Now by the condition (2), we have

$$\left|\int_{-1}^{1}G(\mu_n+t)dt\right|=\int_{-1}^{1}|G(\mu_n+t)|dt=\int_{\mu_n-1}^{\mu_n+1}|G(u)|du,$$

$$\int_{\mu_n-1}^{\mu_n+1}G^2(u)du\leqq|G(v)|\int_{\mu_n-1}^{\mu_n+1}|G(u)|du,$$

where $|G(v)|=\underset{\mu_n-1\leq u\leq\mu_n+1}{\mathrm{Max}}|G(u)|$, which does not exceed $A|G(u)|$ and thus

$$|G(v)|\leqq\frac{A}{2}\int_{\mu_n-1}^{\mu_n+1}|G(u)|du,$$

from which it results that

$$\int_{\mu_n-1}^{\mu_n+1}G^2(u)du\leqq\frac{A}{2}\left(\int_{\mu_n-1}^{\mu_n+1}|G(u)|du\right)^2.$$

By (6) we get

$$(7)\qquad \int_{-\infty}^{\infty}\frac{\sin^2\frac{Lx}{2}}{x^2}\left|\sum_{-N}^{N}\int_{-1}^{1}G(\mu_n+t)e^{-i(\mu_n+t)x}dt\right|^2dx$$

$$\geqq\frac{\pi}{2}(L-2)\frac{2}{A}\sum_{-N}^{N}\left(\int_{\mu_n-1}^{\mu_n+1}G^2(u)du\right).$$

Letting $N\to\infty$, we have

$$(8)\qquad \int_{-\infty}^{\infty}|f(x)|^2\frac{\sin^2\frac{Lx}{2}}{x^2}dx\geqq\frac{\pi L}{2A}\int_{-\infty}^{\infty}G^2(u)du\qquad(L>4).$$

Now by (5)

$$\int_{-\infty}^{\infty}G^2(u)du\geqq2^{-1}\pi^{-2}L\int_{-\frac{\pi}{L}}^{\frac{\pi}{L}}|f(x)|^2dx$$

and in general

$$\int_{-\infty}^{\infty} G^2(u)\,du \geqq 2^{-1}\pi^{-2}L \int_{-\frac{\pi}{L}+y}^{\frac{\pi}{L}+y} |f(x)|^2 dx \tag{9}$$

for all y.

If $C > \frac{\pi}{L}$ and $C > 8A/L$, then by (9)

$$\int_{-C}^{C} |f(x)|^2 dx \leqq 2\pi C \int_{-\infty}^{\infty} G^2(u)\,du\,.$$

Thus

$$\int_{C}^{\infty} + \int_{-\infty}^{C} \frac{|f(x)|^2}{x^2} dx = \left| -\frac{1}{C^2}\int_{-C}^{C} |f(x)|^2 dx + 2\int_{C}^{\infty} \frac{dx}{x^3} \int_{-x}^{x} |f(\xi)|^2 d\xi \right|$$

$$\leqq \frac{2\pi}{C}\int_{-\infty}^{\infty} G^2(u)\,du + \frac{2\pi}{C}\int_{-\infty}^{\infty} |f(x)|^2 dx$$

$$= \frac{2\pi}{C}\int_{-\infty}^{\infty} G^2(u)\,du + \frac{2\pi}{C}\int_{-\infty}^{\infty} G^2(u)\,du$$

$$= \frac{4\pi}{C}\int_{-\infty}^{\infty} G^2(u)\,du\,.$$

Hence from (8)

$$\int_{-C}^{C} |f(x)|^2 \frac{\sin^2 \frac{Lx}{2}}{x^2} dx \geqq \left[\frac{\pi L}{2A} - \frac{4\pi}{C}\right] \int_{-\infty}^{\infty} G^2(u)\,du\,.$$

Now let $C_1 = 8A + \pi$. Then

$$\int_{-\frac{C_1}{L}}^{\frac{C_1}{L}} |f(x)|^2 dx \geqq \frac{4}{L^2} \int_{-\frac{C_1}{L}}^{\frac{C_1}{L}} |f(x)|^2 \frac{\sin^2 \frac{Lx}{2}}{x^2} dx$$

$$\geqq \frac{2\pi^2}{A(8A+\pi)} \frac{1}{L} \int_{-\infty}^{\infty} G^2(u)\,du$$

$$\geqq \frac{1}{A(8A+\pi)} \int_{-\frac{\pi}{L}+y}^{\frac{\pi}{L}+y} |f(x)|^2 dx$$

by (9), which proves the lemma.

Lemma 2. If $f(x)$ satisfies the condition $\mathfrak{A}$ and $|\mu_n - \mu_{n+1}| > L$ and $G(u) = 0$ for $u < 0$, then $f(x)$ can not be analytic in (a, b) unless it is analytic in $(-\infty, \infty)$, where $a - b \geqq 2C_1/L$, C_1 being the constant in Lemma 1.

This can be proved quite similarly as Paley and Wiener did, using Lemma 1.[1] So that the proof will be left to the reader.

3. *Proof of the theorem.* Let $f(x)$ vanish in (a, b). Take n_0 such that

$$\mu_{n+1}-\mu_n \geqq \frac{2C_1}{b-a}+2, \qquad \text{for all } n \geqq n_0,$$

where C_1 is the constant in Lemma 1. $\varphi(x)=f(x)e^{i(\mu_0+1)x}$ is also zero in (a, b) and it is the Fourier transform of the function $G(u+\mu_0+1)=G_1(u)$. The distances of centers of intervals on the positive real axis in which $G_1(u)$ does not identically vanish are not smaller than $2C_1/(b-a)+2$.

Let

$$H_1(u)=\begin{cases} G_1(u), & \text{for } u>0, \\ 0, & \text{for } u\leqq 0, \end{cases}$$

$$H_2(u)=\begin{cases} 0, & \text{for } u>0, \\ G_1(u), & \text{for } u\leqq 0. \end{cases}$$

Then plainly $G_1(u)=H_1(u)+H_2(u)$. Further let the Fourier transforms of $H_1(u)$ and $H_2(u)$ be $\mathfrak{h}_1(x)$ and $\mathfrak{h}_2(x)$ respectively. Then $\varphi(x)=\mathfrak{h}_1(x)+\mathfrak{h}_2(x)$. Since $H_1(u)$ and $H_2(u)$ are of $L_1(-\infty, \infty)$, $\mathfrak{h}_1(x)$ and $\mathfrak{h}_2(x)$ are continuous. And $\mathfrak{h}_1(x)$ is the boundary function of a function analytic in the lower half-plane by the Hille and Tamarkin's result and satisfies the condition in Lemma 2. $\mathfrak{h}_2(x)$ and hence $-\mathfrak{h}_2(x)$ is also the boundary of a function analytic in the upper half-plane. And on (a, b), $\mathfrak{h}_1(x)=-\mathfrak{h}_2(x)$. Thus $\mathfrak{h}_1(x)$ and $\mathfrak{h}_2(x)$ are analytic on (a, b). By Lemma 2 $\mathfrak{h}_1(x)$ is everywhere analytic. Hence $\mathfrak{h}_1(x)=-\mathfrak{h}_2(x)$ everywhere. Thus $\mathfrak{h}_1(x)+\mathfrak{h}_2(x)=\varphi(x)$ vanish identically in $(-\infty, \infty)$, or $f(x)$ vanishes identically which proves the theorem.

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1) See Paley and Wiener, Fourier transforms in the complex domain, pp. 125–127.