PAPERS COMMUNICATED

60. *On the Contractions of Extensors.*

By Akitsugu KAWAGUCHI.

Research Institute for Geometry, Hokkaido Imperial University, Sapporo.

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It is well known that any function $\Phi(v^i, w_j)$ of the components of any two vectors, one of which v^i is contravariant and the other w_j covariant, is to be the function $\Psi(\rho)$ of only their scalar product $\rho = v^i w_i$, if it is not only a scalar but also invariant in its functional form under every regular point transformation of class ω, that is, $\Phi(v^i, w_j) = \Phi(v^r, w_s)$. This fact shows that there exists essentially one and only one scalar product of two vectors v^i, w_j.

In connection with the extended point transformation

$$x^i = x^i(x^r), \qquad x'^i = \frac{\partial x^i}{\partial x^r} x'^r, \qquad i = 1, 2, \dots, N$$

$$x''^i = \frac{\partial x^i}{\partial x^r} x''^r + \frac{\partial^2 x^i}{\partial x^r \partial x^s} x'^r x'^s, \qquad x^{(M)i} = \frac{\partial x^i}{\partial x^r} x^{(M)r} + \cdots,$$

the extensor due to H. V. Craig[1] is defined, for example, by the relationship

$$T_{s\cdot\ \beta t}^{\ \alpha r} = T_{j\cdot\ \delta k}^{\ \gamma i} X_{(\gamma)i}^{(\alpha)r} X_s^j X_{(\beta)t}^{(\delta)k} \qquad \alpha, \beta, \gamma = 0, 1, \dots, M,$$

where we put

$$X_s^j = \frac{\partial x^j}{\partial x^s}, \qquad X_{(\gamma)i}^{(\alpha)r} = \frac{\partial x^{(\alpha)r}}{\partial x^{(\gamma)i}}.$$

The following relations hold for $X_{(\gamma)i}^{(\alpha)r}$

(1) $$X_{(\beta)i}^{(\alpha)r} X_{(\gamma)s}^{(\beta)i} = \delta_\gamma^\alpha \delta_s^r, \qquad X_{(\gamma)u}^{(\alpha)r} = X_{(\beta)i}^{(\alpha)r} X_{(\gamma)u}^{(\beta)i},$$

(2) $$X_{(\beta)i}^{(\alpha)r} = 0 \qquad \text{for} \quad \alpha < \beta$$
$$= \binom{\alpha}{\beta} X_i^{r(\alpha-\beta)} \quad \text{for} \quad \alpha \geqq \beta.^{2)}$$

Although there is only one kind of contraction $\rho = v^i w_i$ for the ordinary vectors v^i, w_j, for the extensors $V^{\alpha i}, W_{\beta j}$ we have not one but $M+1$ kinds of contraction:

(3) $$\rho^{[\alpha]} = \sum_{\beta=\alpha}^{M} \binom{\beta}{\alpha} V^{\beta-\alpha, i} W_{\beta i}.$$

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1) H. V. Craig, On tensors relative to the extended point transformation, American Journal of Mathematics, **59** (1937), 764-774.

2) $X_i^{r(\alpha-\beta)}$ means $\dfrac{d^{\alpha-\beta}}{dt^{\alpha-\beta}} X_i^r$ and we adopt the notation $F^{(\alpha)} = \dfrac{d^\alpha F}{dt^\alpha}$ through this paper.

Do not more other kinds of contractions exist? This question will be answered by the following theorem.

Theorem 1. Any function $\Phi(V^{ai}, W_{\beta j})$ *of the components of any two extensors* V^{ai}, $W_{\beta j}$, *one of which is contravariant and the other covariant, is the function* $\Phi(\rho^{[a]})$ *of only* $M+1$ *quantities* $\rho^{[a]}$, *if it is not only a scalar but also invariant in its functional form under every regular extended point transformation.*

Proof. We shall proceed to establish this theorem by induction. By virtue of the hypotheses it is true that

$$\Phi(V^{ai}, W_{\beta j}) = \Phi(V^{\gamma r}, W_{\delta s})$$
$$= \Phi(V^{ai} X^{(\gamma)r}_{(a)i}, W_{\beta j} X^{(\beta)j}_{(\delta)s}).$$

Differentiating this equation with respect to $X^{r(M)}_i$, we obtain

$$0 = \frac{\partial \Phi}{\partial V^{Mr}} V^{0i} - \frac{\partial \Phi}{\partial W_{0s}} W_{Mj} X^i_s X^j_r .^{1)}$$

Putting $X^i_s = \delta^i_s$ the last equations become

$$\frac{\partial \Phi}{\partial V^{Mj}} V^{0i} = \frac{\partial \Phi}{\partial W_{0i}} W_{Mj}, \tag{4}$$

which yield

$$\frac{\partial \Phi}{\partial V^{Mj}} = \lambda W_{Mj}, \qquad \frac{\partial \Phi}{\partial W_{0i}} = \lambda V^{0i}. \tag{5}$$

It may be assumed $W_{M1} \neq 0$ without loss of generality, for at least one of W_{Mj} is not equal to zero. Then we have

$$V^{M1} = \frac{1}{W_{M1}} \left(\rho^{[0]} - \sum_{j=2}^{N} V^{Mj} W_{Mj} - \sum_{a=0}^{M-1} V^{ai} W_{ai} \right).^{2)} \tag{6}$$

Substitute the right member of (6) for V^{M1} in Φ and denote it with $\Phi^{[1]}$, then we get by virtue of (5)

$$\frac{\partial \Phi^{[1]}}{\partial V^{Mj}} = -\frac{\partial \Phi}{\partial V^{M1}} \frac{W_{Mj}}{W_{M1}} + \frac{\partial \Phi}{\partial V^{Mj}} = 0, \qquad j \geqq 2. \tag{7}$$

On the other hand it will be seen from (4)

$$\frac{\partial \Phi^{[1]}}{\partial W_{0i}} = -\frac{\partial \Phi}{\partial V^{M1}} \frac{V^{0i}}{W_{M1}} + \frac{\partial \Phi}{\partial W_{0i}} = 0. \tag{8}$$

Relations (7) and (8) show that $\Phi^{[1]}$ is independent of V^{Mi} and W_{0i}.

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1) Differentiating equations (1) with respect to $X^{r(M)}_i$ it follows owing to (2)

$$\delta^0_\gamma \delta^a_M \delta^t_r X^i_s + X^{(a)t}_{(\beta)j} \frac{\partial}{\partial X^{r(M)}_i} X^{(\beta)j}_{(\gamma)s} = 0,$$

hence

$$\frac{\partial}{\partial X^{r(M)}_i} X^{(\beta)j}_{(\gamma)s} = -\delta^0_\gamma X^{(\beta)j}_{(M)r} X^i_s = -\delta^0_\gamma \delta^\beta_M X^j_r X^i_s .$$

2) Capital indices it will be recalled are not summed.

Let $\Phi^{[K]}$ be a scalar dependent upon $\rho^{[\lambda]}$, $V^{\alpha i}$ and $W_{\beta j}$, where $\lambda=0, 1, \dots, K-1$; $\alpha=0, 1, \dots, M-K$; $\beta=K, K+1, \dots, M$, and invariant in its functional form under any regular extended point transformation:

$$\Phi^{[K]}(\rho^{[\lambda]}, V^{\alpha i}, W_{\beta j})=\Phi^{[K]}(\rho^{[\lambda]}, V^{\gamma r}, W_{\delta s}).$$

Differentiating this equation with respect to $X_i^{r(M-K)}$ and putting $X_s^i=\delta_s^i$, we obtain

$$(9)\qquad 0=\frac{\partial\Phi^{[K]}}{\partial V^{M-K,j}}V^{0i}-\binom{M}{K}\frac{\partial\Phi^{[K]}}{\partial W_{Ki}}W_{Mj},$$

because

$$\frac{\partial}{\partial X_i^{r(M-K)}}X_{(\gamma)s}^{(\beta)j}=-X_{(\alpha)t}^{(\beta)j}\binom{\alpha}{K}\delta_M^\alpha\delta_r^t X_{(\gamma)s}^{(K)i}=-\delta_M^\beta\delta_\gamma^K\binom{M}{K}X_r^j X_s^i.$$

From (9) it follows

$$(10)\qquad \frac{\partial\Phi^{[K]}}{\partial V^{M-K,j}}=\mu W_{Mj}.$$

Replace

$$V^{M-K,1}=\binom{M}{K}^{-1}\frac{1}{W_{M1}}\left(\rho^{[K]}-\binom{M}{K}\sum_{j=2}^{N}V^{M-K,j}W_{Mj}-\sum_{\beta=K}^{M-1}V^{\beta-K,i}W_{\beta i}\right)$$

for $V^{M-K,1}$ in $\Phi^{[K]}$ and denote it with $\Phi^{[K+1]}$, then we get according to (10)

$$(11)\qquad \frac{\partial\Phi^{[K+1]}}{\partial V^{M-K,j}}=-\frac{\partial\Phi^{[K]}}{\partial V^{M-K,1}}\frac{W_{Mj}}{W_{M1}}+\frac{\partial\Phi^{[K]}}{\partial V^{M-K,j}}=0,\qquad j\geqq 2.$$

On the other hand we get from (9)

$$(12)\qquad \frac{\partial\Phi^{[K+1]}}{\partial W_{Ki}}=-\frac{\partial\Phi^{[K]}}{\partial V^{M-K,1}}\frac{V^{0i}}{W_{M1}}\binom{M}{K}^{-1}+\frac{\partial\Phi^{[K]}}{\partial W_{Ki}}=0.$$

Relations (11) and (12) show that $\Phi^{[K+1]}$ is independent of $V^{M-K,i}$ and W_{Ki}. Hence $\Phi^{[M]}$ must be independent of V^{0i} and W_{Mi}:

$$\Phi(V^{\alpha i}, W_{\beta j})=\Phi^{[M]}(\rho^{[\alpha]})$$

and our theorem is proved.

This theorem may be generalized as follows:

Theorem 2. Any function $\Phi(\underset{\lambda}{V}^{\alpha i}, \underset{\omega}{W}_{\beta j})$ *of the components of any* $H(\leqq N)$ *contravariant extensors* $\underset{\lambda}{V}^{\alpha i}$ $(\lambda=1, 2, \dots, H)$ *and of any* $L(\leqq N)$ *covariant extensors* $\underset{\omega}{W}_{\beta j}$ $(\omega=1, 2, \dots, L)$ *must be dependent upon only* $HL(M+1)$ *scalars*

$$\rho_{\lambda\omega}^{[\alpha]}=\sum_{\beta=\alpha}^{M}\binom{\beta}{\alpha}\underset{\lambda}{V}^{\beta-\alpha,i}\underset{\omega}{W}_{\beta i},$$

whenever it is a scalar and also invariant in its functional form under any regular extended point transformation.

Proof. This theorem can be proven in a similar way as in Theorem 1. That is, in the place of (9) the following relations stand:

$$0=\frac{\partial \Phi^{[K]}}{\partial \underset{\lambda}{V}^{M-K,j}}\underset{\lambda}{V}^{0i}-\binom{M}{K}\frac{\partial \Phi^{[K]}}{\partial \underset{\omega}{W}_{Ki}}\underset{\omega}{W}_{Mj}. \tag{13}$$

$\Phi^{[K]}$ is a scalar dependent on only $\rho^{[\varkappa]}_{\lambda\omega}$, $\underset{\lambda}{V}^{\alpha i}$ and $\underset{\omega}{W}_{\beta j}$ for $\varkappa \leqq K-1$, $\alpha \leqq M-K$ and $\beta \geqq K$. It may be assumed without loss of generality the L ordinary vectors $\underset{\omega}{W}_{Mj}$ be linearly independent and the determinant $W=|\underset{\omega}{W}_{Mj}|$, $\omega, j=1, 2, \ldots, L$, does not vanish, for L extensors $\underset{\omega}{W}_{\beta j}$ may be taken arbitrarily. Likewise, we may assume also that the H ordinary vectors $\underset{\lambda}{V}^{0i}$ are linearly independent. Then $\partial\Phi^{[K]}/\partial \underset{\lambda}{V}^{M-K,j}$ must be expressed in terms of $\underset{\omega}{W}_{Mj}$ linearly:

$$\frac{\partial \Phi^{[K]}}{\partial \underset{\lambda}{V}^{M-K,j}}=\mu^{\lambda\omega}\underset{\omega}{W}_{Mj}. \tag{14}$$

Were it not so and the ordinary vector

$$\overset{\lambda}{X}_j=\frac{\partial \Phi^{[K]}}{\partial \underset{\lambda}{V}^{M-K,j}}-\mu^{\lambda\omega}\underset{\omega}{W}_{Mj}\ ^{1)}$$

were linearly independent of $\underset{\omega}{W}_{Mj}$, we should obtain $\overset{\lambda}{X}_j\underset{\lambda}{V}^{0i}=0$, which contradicts the linear independency of $\underset{\lambda}{V}^{0i}$.

Solving $\underset{\lambda}{V}^{M-K,\omega}$ from the system of equations

$$\rho^{[K]}_{\lambda\omega}=\sum_{\beta=K}^{M}\binom{\beta}{K}\underset{\lambda}{V}^{\beta-K,i}\underset{\omega}{W}_{\beta i},$$

it is obtained

$$\underset{\lambda}{V}^{M-K,\omega}=\binom{M}{K}^{-1}\frac{\overset{\varkappa}{W}^{\omega}}{W}\left(\rho^{[K]}_{\lambda\varkappa}-\sum_{i=L}^{M}\binom{M}{K}\underset{\lambda}{V}^{M-K,i}\underset{\varkappa}{W}_{Mi}-\sum_{\beta=K}^{M-1}\binom{\beta}{K}\underset{\lambda}{V}^{\beta-K,i}\underset{\varkappa}{W}_{Mi}\right), \tag{15}$$

where $\overset{\varkappa}{W}^{\omega}$ is the algebraic complement of the element $\underset{\varkappa}{W}_{M\omega}$ in the determinant W. Replace $\underset{\lambda}{V}^{M-K,\omega}$ in $\Phi^{[K]}$ by the right member of (15) and denote it with $\Phi^{[K+1]}$, then we have owing to (14)

$$\begin{aligned}\frac{\partial \Phi^{[K+1]}}{\partial \underset{\lambda}{V}^{M-K,j}}&=-\frac{\partial \Phi^{[K]}}{\partial \underset{\lambda}{V}^{M-K,\omega}}\frac{\overset{\varkappa}{W}^{\omega}}{W}\underset{\varkappa}{W}_{Mj}+\frac{\partial \Phi^{[K]}}{\partial \underset{\lambda}{V}^{M-K,j}}\\&=-\mu^{\lambda\tau}\underset{\tau}{W}_{M\omega}\frac{\overset{\varkappa}{W}^{\omega}}{W}\underset{\varkappa}{W}_{Mj}+\mu^{\lambda\tau}\underset{\tau}{W}_{Mj}=0,\end{aligned}$$

as $\underset{\tau}{W}_{M\omega}\overset{\varkappa}{W}^{\omega}=\delta^{\varkappa}_{\tau}W$. Moreover it follows from (13)

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1) $\partial\Phi^{[K]}/\partial \underset{\lambda}{V}^{M-K,j}$ is an ordinary covariant vector as can be easily seen.

$$\frac{\partial \Phi^{[K+1]}}{\partial \underset{\omega}{W}^{Ki}} = -\frac{\partial \Phi^{[K]}}{\partial \underset{\lambda}{V}^{M-K,\tau}} \underset{\lambda}{V}^{0i} \binom{M}{K}^{-1} \frac{\overset{\omega}{W}^{\tau}}{W} + \frac{\partial \Phi^{[K]}}{\partial \underset{\omega}{W}^{Ki}}$$

$$= -\frac{\partial \Phi^{[K]}}{\partial \underset{\varkappa}{W}^{Ki}} \underset{\varkappa}{W}_{M\tau} \frac{\overset{\omega}{W}^{\tau}}{W} + \frac{\partial \Phi^{[K]}}{\partial \underset{\omega}{W}^{Ki}} = 0 .$$

Hence $\Phi^{[K+1]}$ does not contain $\underset{\lambda}{V}^{M-K,i}$ and $\underset{\omega}{W}_{Kj}$.

Repeating this method the result is at last attained

$$\Phi(\underset{\lambda}{V}^{\alpha i}, \underset{\omega}{W}_{\beta j}) = \Phi^{[M]}(\rho^{[\alpha]}) .$$

Consequently, the theorem is established.

Quite similarly we can prove

Theorem 3. Any ordinary tensor $T(\underset{\lambda}{V}^{\alpha i}, \underset{\omega}{W}_{\beta j})$, *whose components are dependent on the components of any* $H(\leqq N)$ *contravariant extensors* $\underset{\lambda}{V}^{\alpha i}$ *and of any* $L(\leqq N)$ *covariant extensors* $\underset{\omega}{W}_{\beta j}$, *must be dependent upon only HLM scalars* $\rho^{[\alpha]}_{\lambda\omega}$, $\alpha = 0, 1, \dots, M-1$, *and the ordinary vectors* $\underset{\lambda}{V}^{0i}$ *and* $\underset{\omega}{W}_{Mj}$, *when T is invariant in its functional form under every regular extended point transformation.*

Theorem 2 and Theorem 3 hold also, when $V_{k}^{\cdot\alpha i}$ (or $V^{k\alpha i}$) and $W^{k}_{\cdot\beta j}$ (or $W_{k\beta j}$) stand for $\underset{\lambda}{V}^{\alpha i}$ and $\underset{\omega}{W}_{\beta j}$ respectively.