PAPERS COMMUNICATED

90. *Integral Operator with Bounded Kernel.*

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§ **1.** Let $K(x,y)$ be bounded and measurable in the square $0 \leqq x \leqq 1$, $0 \leqq y \leqq 1$. Consider the integral operator K which transforms the Banach space (L)[1] in (L).

$$(1) \qquad f \to Kf = g: \quad g(y) = \int_0^1 f(x)\, K(x,y)\, dx\,.$$

It is to be noted that such an operator is not always completely continuous[2] in (L). This may be shown by an example (§ 3). We can, however, prove the following

Theorem 1. Let $N(x,y)$ and $K(x,y)$ be bounded and measurable in $0 \leqq x \leqq 1$, $0 \leqq y \leqq 1$. Then the integral operator P defined by the bounded Kernel $P(x,y) = \int_0^1 N(x,z)\, K(z,y)\, dz$ is completely continuous as an operator which maps (L) in (L).

Remark. The integral operator (1) may also be considered as a linear operator which maps (L) in (M),[3] (M) in (M) or (M) in (L).

Proof of Theorem 1: Denote by N and K the integral operators which correspond to the kernels $N(x,y)$ and $K(x,y)$ respectively. P may be considered as a combination of two operators N and K performed successively in this order, where N is an operator which maps (L) in (M) and K is the one which maps (M) in (L): $f \in (L) \to Nf = g \in (M) \to Kg(=Pf) = h \in (L)$.

The unit sphere $\|f\|_L \leqq 1$ of (L) is mapped by N on a set contained in the sphere $\|g\|_M \leqq n$ of (M), where $n = \underset{0 \leq x,y \leq 1}{\text{l. u. b.}} |N(x,y)|$. Hence it is sufficient to prove the

Theorem 2. The integral operator K with bounded kernel $K(x,y)$ is completely continuous as an operator which maps (M) in (L).

Proof: We extend the definition domain of $K(x,y)$ to the infinite square $-\infty < x < +\infty$, $-\infty < y < +\infty$, by putting $K(x,y) = 0$ if the point (x,y) is outside the square $0 \leqq x \leqq 1$, $0 \leqq y \leqq 1$. Let $Kg = h$, where $g \in (M)$, $\|g\|_M \leqq 1$. By Fubuni-Tonelli's theorem, we have

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1) (L) is the space of all the measurable functions $f(x)$ which are absolutely integrable in $0 \leqq x \leqq 1$. For any $f \in (L)$, we define its norm by $\|f\|_L = \int_0^1 |f(x)|\, dx$.

2) A linear operator which maps the Banach space E_1 in another Banach space E_2 is called to be completely continuous if it maps the unit sphere $\|x\| \leqq 1$ of E_1 on a compact (in E_2) set of E_2.

3) (M) is the space of all the bounded measurable functions defined in $0 \leqq x \leqq 1$. For any $f \in (M)$ we define its norm by $\|f\|_M = \underset{0 \leq x \leq 1}{\text{ess. max.}} |f(x)|$.

$$\int_{-\infty}^{\infty}|h(y+\delta)-h(y)|\,dy=\int_{-\infty}^{\infty}dy\left|\int_{-\infty}^{\infty}g(x)\big(K(x,y+\delta)-K(x,y)\big)\,dx\right|$$

$$\leqq\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|K(x,y+\delta)-K(x,y)|\,dxdy\,.$$

The last integral (which is independent of the particular choice of $g\in(M)$, with $\|g\|_M\leqq 1$) tends to zero as δ tends to zero, by Lebesgue's theorem. Further we have $|h(y)|\leqq k$ for any $y(0\leqq y\leqq 1)$ and for any $g\in(M)$ with $\|g\|_M\leqq 1$, where $k=\underset{0\leq x,\,y\leq 1}{\text{l. u. b.}}|K(x,y)|$. Thus by Kolmogoroff-Riesz's theorem[1] the image by K of the unit sphere $\|g\|_M\leqq 1$ of (M) is compact $\big($in $(L)\big)$ in the topology of (L).

Remark. Another proof given below does not appeal to Kolmogoroff-Riesz's theorem.

Consider P as a succession of two operators N and K, each transforming (L) in (L). N is weakly completely continuous as an operator which maps (L) in (L); viz. N maps the unit sphere $\|f\|_L\leqq 1$ of (L) on a set of (L) which is weakly compact in (L).[2] Thus any sequence $\{f_n\}$ with $\|f_n\|_L\leqq 1$ contains a subsequence $\{f_{n_\nu}\}$ such that the sequence $\{g_{n_\nu}\}$ $(g_{n_\nu}=Nf_{n_\nu})$ converges weakly to some element (function) g_0 of (L). As the conjugate space of (L) is (M), we have, by the boundedness of $K(x,y)$

$$\lim_{\nu\to\infty}\int_0^1 g_{n_\nu}(x)\,K(x,y)\,dx=\int_0 g_0(x)\,K(x,y)\,dx \qquad \text{for any } y(0\leqq y\leqq 1).$$

We have $\underset{0\leq y\leq 1}{\text{l. u. b.}}|h_{n_\nu}(y)|\leqq n\cdot k$, $h_{n_\nu}(y)=\int_0^1 g_{n_\nu}(x)\,K(x,y)\,dx$. Hence by Lebesgue's theorem, the sequence $\{h_{n_\nu}(y)\}$ must converge to $h_0(y)=\int_0^1 g_0(x)\,K(x,y)\,dx$ in the strong topology of (L).

§ 2. Let $(\mathfrak{M})$ denote the Banach space consisting of all the set functions $\varphi(E)$, which are completely additive for Borel set of the interval $0\leqq x\leqq 1$. For any $\varphi\in(\mathfrak{M})$ the norm of φ is given by $\|\varphi\|=$ total variation of $\varphi(E)$ in $0\leqq x\leqq 1$. A bounded measurable kernel $K(x,y)$ defines an integral operator which maps $(\mathfrak{M})$ in $(\mathfrak{M})$:

$$\varphi\to K\varphi=\psi:\quad \psi(E)=\int_E dy\int_0^1\varphi(dx)\,K(x,y)\,.$$

Such an operator is not always completely continuous in $(\mathfrak{M})$. The example for the space (L) shows this fact. Corresponding to Theorem 1 we may give the

1) A. Kolmogoroff: Über Kompaktheit der Funktionenmengen bei der Konvergenz im Mittel, Nachr. Ges. Wiss. Göttg., Math.-phys. Kl. 1931, 60–63.

M. Riesz: Sur les ensembles compacts de fonctions sommables, Acta Litt. Sci. Szeged, **6** (1933), 136–142.

2) K. Yosida and S. Kakutani: Applications of Mean Ergodic Theorem to the problem of Markoff's process, Proc. **14** (1938), 333.

Theorem 3. Let $N(x,y)$ and $K(x,y)$ be bounded and measurable in $0 \leqq x \leqq 1,\ 0 \leqq y \leqq 1$. Then the integral operator P defined by the bounded kernel $P(x,y)=\int_0^1 N(x,z)\,K(z,y)dz$ is completely continuous as an operator which maps $(\mathfrak{M})$ in $(\mathfrak{M})$.

Proof: This may be carried out as in the case of Theorem 1. Firstly, N may be considered as a bounded linear operator which maps $(\mathfrak{M})$ in (M):

$$\varphi \rightarrow K\varphi = g: \quad g(y)=\int_0^1 \varphi(dx)\,K(x,y).$$

Secondly, K is a completely continuous linear operator which maps (M) in $(\mathfrak{M})$:

$$g \rightarrow Kg=\psi: \quad \psi(E)=\int_E dy \int_0^1 g(x)\,K(x,y)\,dx.$$

Remark. Another proof analogous to the remark above is also possible. Firstly, N is weakly completely continuous as an operator which maps $(\mathfrak{M})$ in (L). Secondly K is a linear operator which maps the weakly convergent sequence of (L) into the strongly convergent sequence of $(\mathfrak{M})$.

§ **3.** *Example.* We shall construct in this chapter a bounded measurable kernel $K(x,y)$ defined in $0 \leqq x \leqq 1,\ 0 \leqq y \leqq 1$, such that the corresponding integral operator is not completely continuous as an operator which maps (L) in (L).

Put

$$K(x,y)=2 \quad \text{if} \quad \frac{1}{2^n} < x \leqq \frac{1}{2^{n-1}}, \quad \frac{2k}{2^n} < y \leqq \frac{2k+1}{2^n},$$
$$k=0,1,\dots,2^{n-1}-1,$$
$$K(x,y)=0 \quad \text{if} \quad \frac{1}{2^n} < x \leqq \frac{1}{2^{n-1}}, \quad \frac{2k+1}{2^n} < y \leqq \frac{2k+2}{2^n},$$
$$k=0,1,\dots,2^{n-1}-1,$$
$$n=1,2,\dots.$$

$K(x,y)$ is defined in $0 < x \leqq 1,\ 0 < y \leqq 1$. At the points where $x=0$ or $y=0$ put $K(x,y)=1$. This $K(x,y)$ is a required one. It is clear that K is bounded and measurable. In order to prove that the corresponding integral operator is not completely continuous, put

$$f_n(x)=2^n \quad \text{if} \quad \frac{1}{2^n} < x \leqq \frac{1}{2^{n-1}}$$
$$=0 \quad \text{elsewhere in } 0 \leqq x \leqq 1.$$

Then we have $\|f_n\|_L=1$ $(n=1,2,\dots)$ and $\{g_n\}$ $\left(g_n(y)=\int_0^1 f_n(x)\,K(x,y)\,dx\right)$ is not compact in (L) in the topology of (L). For we have, by easy calculation, $\|g_m-g_n\|_L=1$ for any $m \neq n$.

The importance of this example consists in the fact that $K(x,y)$

may be considered as a density of transition probability of a simple Markoff's process. (See the following paper of K. Yosida: Operator-theoretical Treatment of the Markoff's Process.)[1]

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1) After the present paper was completed, we found that Theorem 1 was already obtained by J. Sirvint in another way. *Cf.* J. Sirvint: Sur les transformations intégrales de l'espace L, C. R. URSS. **18** (1938), 255–257. He also obtained an example of a bounded and measurable kernel of the said property. His example coincides with ours up to an additive constant 1.