9. *Boundedness of the Spectrum of a Distribution Function.*

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It is generally recognized that the smoothness of a distribution function $\sigma(x)$ depends on the order of vanishing at infinity of its characteristic function (Fourier-Stieltjes transform of $\sigma(x)$)

$$\Lambda(t;\sigma)=\int_{-\infty}^{\infty}e^{itx}d\sigma(x)\,. \tag{1}$$

For example, if

$$\Lambda(t;\sigma)=O\left(e^{-c|t|}\right),\qquad c \text{ constant}, \tag{2}$$

then $\sigma(x)$ is analytic. Especially it results that $\sigma(x)$ can not be constant in any interval or in other words the spectrum[1] of $\sigma(x)$ becomes the whole real axis. Connecting with the spectrum the following more general fact will be obtained.

Let $p(t)$ be defined in $(0, \infty)$ *and positive and let* $tp'(t)\uparrow\infty$. *If*

$$\Lambda(t;\sigma)=O\left(e^{-p(|t|)}\right) \tag{3}$$

and

$$\int_{1}^{\infty}\frac{p(t)}{t^2}dt=\infty\,, \tag{4}$$

then the spectrum of $\sigma(x)$ is the whole line.

This is known in the case where $\sigma(x)$ is absolutely continuous.[2] In fact, $\sigma'(x)$ becomes indefinitely differentiable and quasi-analytic in the sense of Carlemann-Denjoy. The proof of the above theorem will be done also quite similarly.

Now our purpose is to prove the following fact concerning the converse problem.

Let $p(t)$ be positive and increasing. If

$$\int_{1}^{\infty}\frac{p(t)}{t^2}dt<\infty\,, \tag{5}$$

then there exists a distribution function $\sigma(x)$ with bounded spectrum and such that

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1) The point x is said after A. Wintner the spectrum of a distribution function $\sigma(x)$ if there exist two points x' and x'' in any vicinity of x such that $\sigma(x')\neq\sigma(x'')$.

2) In the case where $\sigma(x)$ is absolutely continuous and $\sigma'(x)$ is periodic and the Fourier coefficients satisfy (3), this was proved by Mandelbrojt. Mandelbrojt, Série de Fourier et classes quasi-analytiques, Borel collections, 1936. And in the case of usual Fourier transforms, see:

Ingham, Notes on Fourier transforms, Journal of London Math. Soc., **9** (1934).

Izumi-Kawata, Tohoku Math. Journal, **42** (1937).

S. Takenaka, ibid. **44** (1938).

(6) $$\Lambda(t;\sigma)=O\,(e^{-p(|t|)})\,.$$

Consider the function

$$q(t)=\int_1^t \frac{p(u)}{u}\,du\,, \qquad \text{for} \quad t>1\,,$$
$$=0\,, \qquad \text{for} \quad 0\leqq t\leqq 1\,,$$

which evidently is strictly increasing for $t>1$. Since

$$\int_t^\infty \frac{p(u)}{u^2}\,du \geqq p(t)\int_t^\infty \frac{du}{u^2}=\frac{p(t)}{t}\,,$$

$p(t)=o\,(t)$ as $t\to\infty$. We may suppose that $p(t)\uparrow\infty$ and that $\int_1^t \frac{p(t)}{t}\,dt\to\infty$, otherwise we may consider the function $p(t)+\sqrt{t}$ in place of $p(t)$.

Now we have

(7) $$\int_1^\infty \frac{q(t)}{t^2}\,dt<\infty\,,$$

which is almost trivial, for

$$\int_1^t \frac{q(u)}{u^2}\,du=q(1)-\frac{q(t)}{t}+\int_1^t \frac{p(u)}{u^2}\,du$$

and further

$$\frac{q(t)}{t}=\frac{1}{t}\int_1^t \frac{p(u)}{u}\,du=\frac{1}{t}\int_1^t o(1)\,du=o(1)\,.$$

Further we have

(8) $$q(3t)-q(t)\geqq p(t)+\log\left(\cos\frac{1}{3}\right)^{-1}, \qquad \text{for} \quad t\geqq t_0\,.$$

For,

$$q(3t)-q(t)=\int_t^{3t}\frac{p(u)}{u}\,du$$
$$\geqq p(t)\log 3\geqq p(t)+\log\left(\cos\frac{1}{3}\right)^{-1}, \qquad \text{for} \quad t\geqq t_0\,.$$

Let $r(t)$ be the inverse function of $q(t)$ and let

$$\varphi(t)=\{r(t)\}^{-1}\,.$$

Then $\varphi(t)\to 0$, for $q(t)\to\infty$. Since

$$t\varphi(t)=\frac{t}{r(t)}=\frac{q\big(r(t)\big)}{r(t)}=o(1)\,,$$

we have

$$\int_1^t \varphi(u)\,du=t\varphi(t)-\varphi(1)-\int_1^t u\varphi'(u)\,du$$
$$=o(1)-\varphi(1)-\int_1^t u\varphi^2(u)\,\frac{\varphi'(u)}{\varphi^2(u)}\,du$$
$$=o(1)-\varphi(1)+\int_{1/\varphi(1)}^{1/\varphi(t)}\frac{q(v)}{v^2}\,dv\,,$$

$\left(\text{putting } \frac{1}{\varphi(u)}=v\right)$ and thus $\int_1^\infty \varphi(u)du<\infty$ from which, by the monotoneness of $\varphi(u)$, it follows that

$$\text{(9)} \qquad \sum_{n=1}^{\infty}\varphi\left(n\log\left(\cos\frac{1}{3}\right)^{-1}\right)=A<\infty\,.$$

Now we put

$$a_n=\varphi\left(n\log\left(\cos\frac{1}{3}\right)^{-1}\right)$$

and define $\sigma(x)$ by the relation

$$\text{(10)} \qquad \prod_{n=1}^{\infty}\cos(a_n t)=\int_{-\infty}^{\infty}e^{itx}d\sigma(x)=\Lambda(t;\sigma)\,.$$

That such $\sigma(x)$ exists and is a distribution function with the spectrum $(-A, A)$ is well known.[1] Thus for our purpose it is sufficient to show (6).

The number of n such that

$$a_n t\geqq c$$

is that of

$$r\left(n\log\left(\cos\frac{1}{3}\right)^{-1}\right)\leqq\frac{t}{c}$$

or that of

$$n\leqq\left\{\log\left(\cos\frac{1}{3}\right)^{-1}\right\}^{-1}q\left(\frac{t}{c}\right)$$

or is

$$\left[\left\{\log\left(\cos\frac{1}{3}\right)^{-1}\right\}^{-1}q\left(\frac{t}{c}\right)\right].^{2)}$$

Hence the number of n such that

$$1>a_n t\geqq\frac{1}{3}$$

is

$$\left[\left\{\log\left(\cos\frac{1}{3}\right)^{-1}\right\}^{-1}q(3t)\right]-\left[\left\{\log\left(\cos\frac{1}{3}\right)^{-1}\right\}^{-1}q(t)\right]$$

$$\geqq\left\{\log\left(\cos\frac{1}{3}\right)^{-1}\right\}^{-1}\left\{q(3t)-q(t)\right\}-1\geqq\left\{\log\left(\cos\frac{1}{3}\right)^{-1}\right\}^{-1}p(t)\,,$$

$$\text{for } t\geqq t_0\,,$$

by (8).

Hence

$$|\Lambda(t;\sigma)|=\left|\prod_{n=1}^{\infty}\cos(a_n t)\right|\leqq\prod_{1>a_n t\geqq\frac{1}{3}}\cos(a_n t)$$

$$\leqq\left(\cos\frac{1}{3}\right)^{\{\log(\cos\frac{1}{3})^{-1}\}^{-1}p(t)}=e^{-p(t)}.$$

This completes the proof.

1) B. Jessen and A. Wintner, Distribution functions and the Riemann zeta function. Transaction Amer. Math. Soc. **38** (1935).

2) $[x]$ represents the integral part of x.