PAPERS COMMUNICATED

7. *On the Generalized Circles in the Conformally Connected Manifold.*

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As in Mr. K. Yano's paper[1] in which the same problem is studied, take in the tangential space an $(n+2)-$ spherical "repère naturel" $[A_P]$ satisfying the following equations[2]:

$$A_0^2=A_\infty^2=A_0A_i=A_\infty A_j=0\,,\qquad A_0A_\infty=-1\,,\qquad A_iA_j=G_{ij}=\frac{g_{ij}}{g^{\frac{1}{n}}}\,, \tag{1}$$
$$(i, j, k, \dots=1, 2, \dots, n)$$

the connection being defined by

$$dA_P=\omega_P^Q A_Q\,,\qquad (P, Q, R, \dots=0, 1, \dots, n, \infty) \tag{2}$$

where
$$\omega_P^Q=\Pi_{Pk}^Q dx^k\,, \tag{3}$$

$$\left.\begin{aligned} &\Pi_{0k}^\infty=\Pi_{\infty k}^0=\Pi_{0k}^0=\Pi_{\infty k}^\infty=0\,,\quad \Pi_{0j}^i=\delta_j^i\,,\quad \Pi_{jk}^\infty=G_{jk}\,,\quad G_{ij}\Pi_{\infty k}^j=\Pi_{jk}^0 \\ &\Pi_{jk}^i=\frac{1}{2}G^{ih}(\partial_j G_{kh}+\partial_k G_{jh}-\partial_h G_{jk}) \end{aligned}\right\} \tag{4}$$

Then any curve $x^i(s)$ in the manifold can be developed into a curve in the tangential space at any point $x^i(s_0)$ on the curve by the formulae (2). Let us consider the curves whose developments are circles.

When we take two quantities a^P and b^P which are contragradient to A_P and satisfy the equations

$$\left.\begin{aligned} &G_{PQ}a^Pa^Q=1\,,\qquad G_{PQ}a^Pb^Q=0\,,\qquad G_{PQ}b^Pb^Q=0\,, \\ &a^\infty=0\,, \end{aligned}\right\} \tag{5}$$

where
$$G_{PQ}=A_PA_Q\,,$$

then
$$\frac{1}{b^\infty}A_0+a^\alpha A_\alpha t+\frac{1}{2}b^PA_Pt^2 \qquad (\alpha=0, 1, 2, \dots, n) \tag{6}$$

is an invariant and represents a circle in the tangential space. Because of (5), (6) becomes, when multiplied by b^∞,

$$\begin{aligned} A&=A_0+b^\infty a^\alpha A_\alpha+\frac{1}{2}b^\infty b^PA_Pt^2 \\ &=\Big(1+G_{ij}a^ib^jt+\frac{1}{4}G_{ij}b^ib^jt^2\Big)A_0+\Big(b^\infty a^it+\frac{1}{2}b^\infty b^it^2\Big)A_i+\frac{1}{2}(b^\infty t)^2A_\infty\,. \end{aligned} \tag{7}$$

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1) K. Yano: Sur les circonférences généralisées dans les espaces à connexion conforme, Proc. **14** (1938), 329-32.

2) K. Yano: Remarques relatives à la théorie des espaces à connexion conforme, Comptes Rendus, **206** (1938), 560-2.

When the development of the curve is a circle, the equation

$$\frac{dA}{ds}=\alpha A \tag{8}$$

must be satisfied along this curve for suitably chosen a^P, b^P and α. This equation must hold for any value of t but α and $\frac{dt}{ds}$ may contain t.

From (7) and (8) we obtain

$$\frac{d}{ds}\Big(G_{ij}a^ib^jt+\frac{1}{4}G_{ij}b^ib^jt^2\Big)+\Big(b^\infty a^it+\frac{1}{2}b^\infty b^it^2\Big)\Pi^0_{ik}x'_k$$
$$=\alpha\Big(1+G_{ij}a^ib^jt+\frac{1}{4}G_{ij}b^ib^jt^2\Big), \tag{9, a}$$

$$\frac{d}{ds}\Big(b^\infty a^it+\frac{1}{2}b^\infty b^it^2\Big)+\Big(1+G_{jk}a^jb^kt+\frac{1}{4}G_{jk}b^jb^kt^2\Big)x'^i$$
$$+\Big(b^\infty a^jt+\frac{1}{2}b^\infty b^jt^2\Big)\Pi^i_{jk}x'^k+\frac{1}{2}(b^\infty t)^2\Pi^i_{\infty k}x'^k=\alpha\Big(b^\infty a^it+\frac{1}{2}b^\infty b^it^2\Big), \tag{9, b}$$

$$\frac{1}{2}\frac{d}{ds}(b^\infty t)^2+G_{ij}\Big(b^\infty a^it+\frac{1}{2}b^\infty b^it^2\Big)x'^j=\alpha\frac{1}{2}(b^\infty t)^2, \tag{9, c}$$

where
$$x'^i=\frac{dx^i}{ds}.$$

As t is an invariant in (6) it is expected that an invariant parameter s is obtained by putting $\frac{dt}{ds}=1$ in the equations (9). Expanding α in series

$$\alpha=\alpha_0+\alpha_1t+\alpha_2t^2+\cdots$$

and comparing the coefficients of t^n's in (9) we get from (9, a)

$$\alpha_0=G_{ij}a^ib^j,$$

and from (9, c), $\alpha_i=0$ for $i\neq 0$,

and because of these, (9) becomes

$$\frac{d}{ds}(G_{ij}a^ib^j)+\frac{1}{2}G_{ij}b^ib^j+b^\infty a^i\Pi^0_{ik}x'^k-(G_{ij}a^ib^j)^2=0, \tag{10, a}$$

$$\frac{d}{ds}(G_{ij}b^ib^j)+2\Pi^0_{jk}b^\infty b^jx'^k-G_{ij}a^ib^jG_{lm}b^lb^m=0, \tag{10, b}$$

$$b^\infty a^i+x'^i=0, \tag{11, a}$$

$$\frac{d}{ds}(b^\infty a^i)+\Pi^i_{jk}b^\infty a^jx'^k+b^\infty b^i+G_{jk}a^jb^k(x'^i-b^\infty a^i)=0, \tag{11, b}$$

$$\frac{d}{ds}(b^\infty b^i)+\Pi^i_{jk}b^\infty b^jx'^k+\frac{1}{2}G_{jk}b^jb^kx'^i+(b^\infty)^2\Pi^i_{\infty k}x'^k-b^\infty b^iG_{jk}a^jb^k=0, \tag{11, c}$$

$$(b^\infty)^2+G_{ij}b^\infty a^ix'^j=0, \tag{12. a}$$

$$\frac{d}{ds}(b^\infty)^2+G_{ij}b^\infty a^ix'^j-(b^\infty)^2G_{ij}a^ib^j=0, \tag{12, b}$$

From (11, a) we get because of (5), that is, $G_{ij}a^i a^j=1$

$$b^{\infty}=\sqrt{G_{ij}x'^i x'^j}=l. \tag{13}$$

When we define

$$v^i=x''^i+\Pi^i_{jk}x'^j x'^k, \tag{14}$$

(11, b) and (11, a) give us

$$v^i=lb^i+2G_{jk}a^j b^k x'^i,$$

hence

$$l\,l'=G_{jk}a^j b^k l^2,$$

$$G_{jk}a^j b^k=l^{-1}l', \tag{15}$$

and consequently

$$b^i=l^{-1}v^i-2l^{-2}l'x'^i, \qquad G_{jk}b^j b^k=l^{-2}G_{jk}v^j v^k, \tag{16}$$

where l' denotes $\frac{dl}{ds}$.

Then from (10, a) we get

$$l^{-1}l''-2l^{-2}l'^2+\frac{1}{2}l^{-2}G_{jk}v^j v^k-\Pi^0_{jk}x'^j x'^k=0, \tag{17}$$

and from (11, c),

$$\frac{d}{ds}(v^i-2l^{-1}l'x'^i)+\Pi^i_{jk}(v^j-2l^{-1}l'x'^j)x'^k+\frac{1}{2}l^{-2}G_{jk}v^j v^k x'^i$$
$$+l^2\Pi^i_{\infty k}x'^k-l^{-1}l'(v^i-2l^{-1}l'x'^i)=0,$$

which becomes because of (17)

$$\frac{d}{ds}v^i+\Pi^i_{jk}v^j x'^k-3l^{-1}l'v^i+\frac{3}{2}l^{-2}G_{jk}v^j v^k x'^i-2\Pi^0_{jk}x'^j x'^k x'^i$$
$$+l^2\Pi^i_{\infty k}x'^k=0. \tag{18}$$

It will be easily verified that (10, b), (12, a), (12, b), and (17) are all satisfied by (18), which are the equations of the curve sought.

As we put $\frac{dt}{ds}=1$, it is necessary to prove that there is no curve which can not be expressed in the form (18), the development being a circle. This is easily done, because for any given initial values of a^P and b^P satisfying (5) a curve satisfying (18) exists, and every circle in the tangential space passing through the point of contact is expressible in the form (6).

Now (18) are the equations of a generalized circle in the manifold with conformal connection (2), (3), (4) where s is an invariant parameter. When we take another parameter σ which is not invariant under the transformation of coordinates x^i but satisfies the simpler equation

$$G_{ij}\frac{dx^i}{d\sigma}\frac{dx^j}{d\sigma}=1, \tag{19}$$

we have

$$l=\frac{d\sigma}{ds},$$

$$v^i=\left(\frac{d^2x^i}{d\sigma^2}+\Pi^i_{ab}\frac{dx^a}{d\sigma}\frac{dx^b}{d\sigma}\right)\left(\frac{d\sigma}{ds}\right)^2+\frac{dx^i}{d\sigma}\frac{d^2\sigma}{ds^2},$$

$$G_{ij}v^iv^j=G_{ij}\left(\frac{d^2x^i}{d\sigma^2}+\Pi^i_{ab}\frac{dx^a}{d\sigma}\frac{dx^b}{d\sigma}\right)\left(\frac{d^2x^j}{d\sigma^2}+\Pi^j_{cd}\frac{dx^c}{d\sigma}\frac{dx^d}{d\sigma}\right)\left(\frac{d\sigma}{ds}\right)^4+\left(\frac{d^2\sigma}{ds^2}\right)^2,$$

and consequently we get from (17) and (18)

$$\{s\}_\sigma=\frac{1}{2}G_{ij}\left(\frac{d^2x^i}{d\sigma^2}+\Pi^i_{ab}\frac{dx^a}{d\sigma}\frac{dx^b}{d\sigma}\right)\left(\frac{d^2x^j}{d\sigma^2}+\Pi^j_{cd}\frac{dx^c}{d\sigma}\frac{dx^d}{d\sigma}\right)$$
$$-\Pi^0_{ij}\frac{dx^i}{d\sigma}\frac{dx^j}{d\sigma} \qquad (20)$$

and

$$\frac{d}{d\sigma}\left(\frac{d^2x^i}{d\sigma^2}+\Pi^i_{jk}\frac{dx^j}{d\sigma}\frac{dx^k}{d\sigma}\right)+\Pi^i_{jk}\left(\frac{d^2x^j}{d\sigma^2}+\Pi^j_{lm}\frac{dx^l}{d\sigma^2}\frac{dx^m}{d\sigma}\right)\frac{dx^k}{d\sigma}$$
$$+\frac{dx^i}{d\sigma}\left[G_{jk}\left(\frac{d^2x^j}{d\sigma^2}+\Pi^j_{ab}\frac{dx^a}{d\sigma}\frac{dx^b}{d\sigma}\right)\left(\frac{d^2x^k}{d\sigma^2}+\Pi^k_{cd}\frac{dx^c}{d\sigma}\frac{dx^d}{d\sigma}\right)-\Pi^0_{jk}\frac{dx^j}{d\sigma}\frac{dx^k}{d\sigma}\right]$$
$$+\Pi^i_{\infty k}\frac{dx^k}{d\sigma}=0. \qquad (21)$$

These are just the same expressions as obtained by Mr. K. Yano when we put $M=1$ and constant$=0$. That these two equations are necessary will be published by him too.