65. *Asymptotic Almost Periodicities and Ergodic Theorems.*

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1. *Introduction.* Two ergodic theorems, the mean ergodic theorem (M. E. T.) and a generalisation of Fréchet-Kryloff-Bogoliouboff's theorem (F-K-B E. T.), were obtained in the preceding notes.[1] There are some strong difference or gap between these two ergodic theorems. The purpose of the present note is to fulfill this gap with new ergodic theorems and to show that these theorems (including the M. E. T. and the F-K-B E. T.) are intimately related to the properties of *asymptotic almost periodicity*, to be defined below.

Let T denote a continuous (bounded) linear operation defined on a Banach space B to B, and consider the sequence $\{T^n \cdot x\}$, $x \in B$, $n = 1, 2, \ldots$. Corresponding to the various assumptions of *total boundedness* of $\{T^n \cdot x\}$, we may obtain various ergodic theorems together with the respective properties of asymptotic almost periodicity (in n) of $T^n \cdot x$. This simple idea was suggested by Bochner-Neumann's theory[2] (B-N theory) of almost periodic functions in groups. However, since we do not assume the existence of the inverse T^{-1} of T, we are here concerned with the *semi-group* of the addition of positive integers. We also remark that a new proof of the existence of the mean for Bohr's (Stepanoff's, Muckenhaupt's and other author's) almost periodic functions may be obtained by virtue of the ergodic theorems. Combined with the Fourier analysis in the B-N theory, the M. E. T. yields a Fourier expansion theorem and a theorem of existence of the proper values for *unitary* (isometric) *operator* T of B. Lastly it is to be noted that the B-N theory also suggests us not to confine ourselves to the Banach spaces; the (ergodic) theorems obtained may be extended to *linear topological spaces.*

2. *Ergodic theorems and asymptotic almost periodicities.*

Theorem 1. We assume that T satisfies the following total boundedness:

$$(1) \qquad \| T^n \| \leq a \text{ constant } C \ (n=1,2,\ldots),$$

$$(2) \qquad \begin{cases} \text{for given } x \in B, \text{ the sequence } \{x_n\},\ x_n = \dfrac{T+T^2+\cdots+T^n}{n} \cdot x \\ (n=1,2,\ldots), \text{ contains a subsequence weakly convergent to a} \\ \text{point } x_0 \in B. \end{cases}$$

Then x_n converges strongly to x_0 and we have $T \cdot x_0 = x_0$. If (2) *is satisfied for all $x \in B$, then x_0 is defined by a continuous linear operation T_1 such that*

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1) Proc. **14** (1938), 286–294. Cf. also S. Kakutani: ibd., 295–298.

2) Trans. Amer. Math. Soc. **37** (1935), 21–50.

$$(3) \qquad TT_1 = T_1T = T_1^2 = T_1 .$$

This is the M. E. T., expressed somewhat more precisely than in the preceding note. T_1 is the projection operator which maps B on the proper space of T belonging to the proper value 1. Since we have $T^n \cdot x = x_0 + (T^n \cdot x - x_0)$, $T^n \cdot x$ is periodic in n (with the period zero) except the error whose arithmetic mean $\frac{1}{m}\sum_{n=k}^{k+m-1}(T^n \cdot x - x_0)$ tends strongly to zero *uniformly in* k when m tends to $+\infty$. If, in particular, T is a unitary operator in the hilbert space, then we have a more precise result concerning the almost periodicity in n of $T^n \cdot x$, i. e. E. Hopf's theorem.[1]

Theorem 2. We assume that T satisfies the condition (1) *and*

$$(4) \qquad \begin{cases} \{T^n \cdot x\} \textit{ is totally bounded in the strong topology defined by} \\ \textit{the norm in } B. \end{cases}$$

Then, since the convex closure of $\{T^n \cdot x\}$ is totally bounded, the M. E. T. of course applies to T. (It is to be noted that, in this case, the proof of the M. E. T. may be shortened; it can be obtained without appealing to the Hahn-Banach's extension theorem.[2]) Moreover, we have the asymptotic almost periodicity (in n) of $T^n \cdot x$:

$$(5) \qquad \begin{cases} \textit{for any } \varepsilon > 0, \textit{ there exists a positive integer } p_\varepsilon \textit{ such that any} \\ \textit{interval of length } p_\varepsilon \textit{ with positive integers as its extremities con-} \\ \textit{tains at least one integer } p \textit{ satisfying } \varlimsup_{n\to\infty} \| T^{n+p} \cdot x - T^n \cdot x \| \leqq \varepsilon. \end{cases}$$

Proof of (5). There exists, by (4), a positive integer p_ε such that

$$\min_{1 \leq k \leq p_\varepsilon} \| T^m \cdot x - T^k \cdot x \| \leqq \frac{\varepsilon}{C} \quad \text{for any } m \ (=1, 2, \dots).$$

Hence, by (1), there exists $k(m)$, $1 \leqq k(m) \leqq p_\varepsilon$, such that

$$\| T^{n+m} \cdot x - T^{n+k(m)} \cdot x \| \leqq \varepsilon \quad \text{for } n = 1, 2, \dots .$$

Thus we have

$$\min_{m-p_\varepsilon \leq p \leq m-1} \{ \varlimsup_{n\to\infty} \| T^{n+p} \cdot x - T^n \cdot x \| \} \leqq \varepsilon \quad \text{for } m = p_\varepsilon + 1, \ p_\varepsilon + 2, \dots . \qquad \text{Q. E. D.[3]}$$

1) E. Hopf: Ergodentheorie, Berlin (1937), 25.

2) For the sake of comprehension, we here sketch the proof for the (strong) convergence of the sequence $\{x_n\}$, $x_n = \frac{T + T^2 + \cdots + T^n}{n} \cdot x$ $(n = 1, 2, \dots)$. Since $\{x_n\}$ is totally bounded, $\{x_n\}$ contains a subsequence $\{x_{n'}\}$ such that $\lim_{n'\to\infty} x_{n'} = x_0 \in B$. Clearly we have $T \cdot x_0 = x_0$, and hence we would obtain $\lim_{n\to\infty} x_n = x_0$ if $\lim_{n\to\infty} \frac{T + T^2 + \cdots + T_n}{n} \cdot (x - x_0) = 0$. Since T^n is of norm $\leqq C$ $(n = 1, 2, \dots)$, this last equation is clear when $(x - x_0)$ is of the form $(y - T \cdot y)$, $y \in B$; and it is also clear when $(x - x_0)$ is a limit element of the form $(y - T \cdot y)$, $y \in B$. Thus from $(x - x_0) = \lim_{n'\to\infty} \left(x - \frac{T + T^2 + \cdots + T^{n'}}{n'} \cdot x \right) = \lim_{n'\to\infty} (E - T) \left(\frac{n'E + (n'-1)T + \cdots + T^{n'}}{n'} \right) \cdot x$, E = the identity operator, we see that $\lim_{n\to\infty} x_n = x_0$.

3) The original proof is somewhat shortened by S. Kakutani's remark.

Example. Let a continuous point transformation P defined on a compactum R to R be a *contraction*:

$$\text{distance } (P\cdot t, P\cdot s) \leqq \text{distance } (t, s) \quad \text{for any} \quad t, s \in R.$$ [1]

Let B denote the Banach space of all the complex-valued continuous functions $f(t)$ on R with the norm $\|f\| = \max_{t\in R} |f(t)|$, then P defines a linear operation T on B to $B: T\cdot f = g$, $g(t) = f(P\cdot t)$. It is easy to see that the conditions (1) and (4) are satisfied for any $x \in B$. Another example is given from the theory of almost periodic functions. See the next paragraph.

The set of all the continuous linear operations T on B to B constitutes a Banach space $\tilde{B}$ with the norm $\|T\|$ $(= \text{l. u. b.}_{\|x\|\leq 1} \|T\cdot x\|)$. Our next theorem reads as follows.

Theorem 3. We assume that T satisfies (1) *and*

(6) $\{T^n\}$ *is totally bounded in the topology defined by the norm in* $\tilde{B}$.

Then the M. E. T. applies to T in the uniform sense:

$$\lim_{n\to\infty} \left\| \frac{T+T^2+\cdots+T^n}{n} - T_1 \right\| = 0. \tag{7}$$

Moreover, we have the asymptotic almost periodicity in the uniform sense, viz. we have, in (5), $\overline{\lim_{n\to\infty}} \|T^{n+p} - T^n\| \leqq \varepsilon$ *instead of* $\overline{\lim_{n\to\infty}} \|T^{n+p}\cdot x - T^n\cdot x\| \leqq \varepsilon$.

We omit the proof.

Theorem 4. We assume, beside (1) *and* (6), *that any limit element* $(\in \tilde{B})$ *of the totally bounded set* $\{T^n\}$ *is a completely continuous linear operation on B to B. This case is precisely the case of the F-K-B E. T.*

Proof. For the proof we have only to restate the F-K-B E. T.:

Let T satisfy (1), and let there exist an integer m and a completely continuous linear operation V on B to B such that $\|T^m - V\| < 1$, then the proper values with modulus 1 of T are finite in number. Let these proper values be $\lambda_1, \lambda_2, \ldots, \lambda_k$, then there exist completely continuous linear operations $T_{\lambda_1}, T_{\lambda_2}, \ldots, T_{\lambda_k}$ and a continuous linear operation S such that

$$\tag{8} \begin{cases} T = \sum_{i=1}^{k} \lambda_i T_{\lambda_i} + S, \quad T_{\lambda_i}^2 = T_{\lambda_i}, \quad T T_{\lambda_i} = T_{\lambda_i} T = \lambda_i T_{\lambda_i}, \\ T_{\lambda_i} T_{\lambda_j} = 0 \quad (i \neq j), \; S T_{\lambda_i} = T_{\lambda_i} S = 0 \quad (i, j = 1, 2, \ldots, k), \\ \|S^n\| \leqq \dfrac{\delta}{(1+\varepsilon)^n} \quad (n = 1, 2, \ldots) \text{ with positive constants } \varepsilon, \delta. \end{cases}$$

Since we obtain $T^n = \sum_{i=1}^{k} \lambda_i^n T_{\lambda_i} + S^n$ $(n = 1, 2, \ldots)$ from (8), the asymptotic almost periodicity in n of T^n is apparent.

3. *A proof of the existence of the mean for Bohr's almost periodic functions.*

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1) It is sufficient to assume that distance $(P^n\cdot t, P^n\cdot s) \leqq$ distance (t, s) multiplied by a constant independent of n.

Theorem 5. Let a complex-valued continuous function $f(t)$ on the infinite interval $(-\infty, \infty)$ be almost periodic in Bohr's sense, then

$$\lim_{u\to\infty}\frac{1}{u}\int_s^{s+u} f(t)\,dt \quad \textit{exists uniformly in } s,\ -\infty<s<\infty\,.$$

Proof. According to S. Bochner and J. Favard, the sequence $\{f^{(n)}(t)\}$, $f^{(n)}(t)=f(t+n)$ $(n=1, 2, \dots)$, is totally bounded in the topology defined by the distance $\|f^{(n)}-f^{(m)}\|=\underset{-\infty<t<\infty}{\text{l. u. b.}}|f^{(n)}(t)-f^{(m)}(t)|$. Let B denote the Banach space spanned by $\{f^{(n)}\}$ with the above norm $\|\ \ \|$, and let T be the continuous linear operation on B to B defined by $T\cdot f=g$, $g(t)=f(t+1)$. T is surely of norm 1 and the Theorem 2 applies to T. Hence the sequence $\{f_n\}$, $f_n=\dfrac{T+T^2+\cdots+T^n}{n}\cdot f$ $(n=1, 2, \dots)$, converges strongly to an element ϵB.[1)]

Hence, we have, uniformly in s $(-\infty<s<\infty)$,

$$\int_0^1\left\{\lim_{n\to\infty}\frac{1}{n}\sum_{m=1}^n f(s+t+m)\right\}dt=\lim_{n\to\infty}\frac{1}{n}\sum_{m=1}^n\int_0^1 f(s+t+m)\,dt$$

$$=\lim_{n\to\infty}\frac{1}{n}\int_1^{n+1} f(s+t)\,dt\,.$$

This proves the theorem, by the uniform boundedness of $f(t)$.

4. *Application of the Fourier analysis to the operator T.* Throughout this paragraph, we assume that the inverse T^{-1} of T exists.

Theorem 6. We assume that

(9) $\|T^n\|\leqq$ *a constant* C $(n=0, \pm1, \pm2, \dots)$,

(10) $\left\{\begin{array}{l}\textit{for any } x\in B, \textit{ the set } \{T^n\cdot x\}\ (n=0, \pm1, \pm2, \dots) \textit{ is totally}\\ \textit{bounded in the strong topology defined by the norm in } B.\end{array}\right.$

Then there exists at least one proper value with modulus 1 *of T.*

Proof. From (9) and (10) we see that the set $\{F(n)\}$, $F(n)=T^n\cdot x$ $(n=0, \pm1, \pm2, \dots)$, is totally bounded by the distance $\|F(n)-F(m)\|=\underset{-\infty<p<\infty}{\text{l. u. b.}}\|T^{n+p}\cdot x-T^{m+p}\cdot x\|$. Hence $F(n)$ is almost periodic in the group of addition of all the integers. Thus, by the Weierstrass approximation theorem in the B-N theory, we obtain the *Fourier expansion*:

(11) $$T^n\cdot x=F(n)\sim\sum_{i=1}^n\lambda_i^n C_{\lambda_i}\quad(n=0, \pm1, \pm2, \dots),\ |\lambda_i|=1\quad(i=1, 2, \dots)\,,$$

where the *Fourier coefficients* C_{λ_i} are given by

(12) $$C_{\lambda_i}=\lim_{n\to\infty}\frac{1}{n}\sum_{m=1}^n\lambda_i^{-m}F(m)=\lim_{n\to\infty}\left(\frac{1}{n}\sum_{m=1}^n\lambda_i^{-m}T^m\right)\cdot x\,.$$

By the M. E. T., $T_{\lambda_i}=\lim\limits_{n\to\infty}\dfrac{1}{n}\sum\limits_{m=1}^n\lambda_i^{-m}T^m$ is the projection operator which

2) Cf. the footnote 2).

maps B on the proper space of T belonging to the proper value λ_i. Hence, if T does not admit proper value with modulus 1, then the Fourier expansion (11) is nought for any $x \in B$. By the uniqueness theorem in the B-N theory, this means that $T^n \cdot x = 0$ for any $x \in B$ and n. Thus, in particular, $T^0 \cdot x = x = 0$ for any $x \in B$, which is surely a contradiction.

Remark. If we put $|||x||| = \underset{-\infty < n < \infty}{\text{l. u. b.}} \|T^n \cdot x\|$, we have $|||T^n \cdot x||| = |||x|||$. This new norm $|||x|||$ gives an equivalent topology as $\|x\|$. Hence, the condition (9) means, in essential, that T is a unitary (=isometric) operator in B.

Corollary. As an application of the theorem we have the following result. Let a one-to-one isometric point transformation P of a compactum R on itself be $\neq$ the identical transformation, then there exist a complex-valued continuous function $f(t)$ on R and a complex number $\lambda \neq 1$, $|\lambda| = 1$, such that $f(P \cdot t) = \lambda f(t)$ for all $t \in R$. In other words, there exists at least one *angle variable* of the transformation P.

The proof of the Theorem 6 suggests the

Theorem 7. We assume (9), (10) *and*

$$(13) \quad \begin{cases} T \textit{ admits at most enumerably infinite number of proper} \\ \textit{values } \{\lambda_i\} \textit{ with modulus } 1, \ (i=1, 2, \ldots), \end{cases}$$

then we have the Fourier expansion:

$$(14) \quad \begin{cases} T^n \sim \sum_{i=1}^{\infty} \lambda_i^n T_{\lambda_i} \ (n=0, \pm 1, \pm 2, \ldots), \ T_{\lambda_i} = \lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} \lambda_i^{-m} T^m, \\ T T_{\lambda_i} = T_{\lambda_i} T = \lambda_i T_{\lambda_i}, \ T_{\lambda_i}^2 = T_{\lambda_i}, \ T_{\lambda_i} T_{\lambda_j} = 0 \ (i \neq j), \ (i, j = 1, 2, \ldots). \end{cases}$$

If the expansion converges in weak or strong sense, then it represents T^n (in weak or strong sense). Moreover we have the Parseval theorem in the weak sense:

$$(15) \qquad \lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} |f(T^m \cdot x)|^2 = \sum_{i=1}^{\infty} |f(T_{\lambda_i} \cdot x)|^2,$$

for any $x \in B$ and for any linear functional f on B.

Thus B is decomposed into the proper spaces $T_{\lambda_i} \cdot B$ belonging to the proper values λ_i with modulus one of T.