PAPERS COMMUNICATED

62. *A Proof of a Theorem of Hardy and Littlewood Concerning Strong Summability of Fourier Series.*

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1. Let $f(x)$ be integrable and periodic with period 2π and let its Fourier series be

$$f(x) \sim \frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) . \tag{1}$$

If $f(x) \in L_p$ $(p > 1)$, then (1) is strongly summable for any positive index at a Lebesgue set, that is:

$$\sum_{\nu=0}^{n} | s_\nu(x) - f(x) |^k = o(n) , \tag{2}$$

for every $k > 0$, where s_ν is the partial sums of (1). If $f(x)$ is merely integrable (2) does not necessarily hold at the Lebesgue set.[1] Professors G. H. Hardy and J. E. Littlewood proved, however, the following theorem.[2]

Theorem. If

$$\int_0^t | \phi(u) | du = o(t) , \tag{3}$$

then

$$\sum_{\nu=0}^{n} | s_\nu(x) - f(x) |^2 = o(n \log n) , \tag{4}$$

where

$$\phi(u) = \frac{1}{2} \{ f(x+u) + f(x-u) - 2f(x) \} . \tag{5}$$

They proved this theorem by power series method. The object of this paper is to give an elementary proof.

2. We make the ordinary simplifications. Suppose that $f(t)$ is even and $x=0$, $f(0)=0$, so that $\phi(u)=f(u)$. Thus we shall prove, under the condition

$$\int_0^t |f(u)| du = \Phi(t) = o(t), \tag{6}$$

that

$$\sum_{\nu=0}^{n} s_\nu^2 = o(n \log n) . \tag{7}$$

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1) This is due to Hardy and Littlewood, The strong summability of Fourier series, Fund. Math., **25** (1935), 162–189.

2) Hardy-Littlewood, loc. cit. It is unsolved, however, whether (2) holds almost everywhere.

We first note that under (6)

$$\int_{1/n}^{\pi} \frac{|f(t)|}{t} dt = o(\log n), \tag{8}$$

$$\int_{1/n}^{\pi} \frac{|f(t)|}{t^2} dt = o(n) \tag{9}$$

which are easily obtained by integration by parts.

Now we have

$$\begin{aligned} \sum_{\nu=0}^{n} s_\nu^2 &= \sum_{\nu=0}^{n} \frac{1}{\pi^2} \int_0^\pi f(t) \frac{\sin \nu t}{t} dt \int_0^\pi f(u) \frac{\sin \nu u}{u} du + o(n) \\ &= \frac{1}{\pi^2} \int_0^\pi \frac{f(t)}{t} dt \int_0^\pi \frac{f(u)}{u} \sum_{\nu=1}^{n} \sin \nu t \sin \nu u + o(n) \\ &= \int_0^{1/n} \int_0^{1/n} + \int_0^{1/n} \int_{1/n}^{\pi} + \int_{1/n}^{\pi} \int_0^{1/n} + \int_{1/n}^{\pi} \int_{1/n}^{\pi} + o(n) \\ &= J_1 + J_2 + J_3 + J_4 + o(n), \end{aligned} \tag{10}$$

say. We have

$$|J_1| \leqq \frac{1}{\pi^2} \int_0^{1/n} |f(t)| dt \int_0^{1/n} |f(u)| \sum_{\nu=1}^{n} \nu^2 = o(n), \tag{11}$$

$$|J_2| \leqq \frac{1}{\pi^2} \int_0^{1/n} |f(t)| dt \int_{1/n}^{\pi} \frac{|f(u)|}{u} \sum_{\nu=1}^{n} \nu du$$

which is, by (6) and (8), less than

$$\begin{aligned} &\frac{n^2}{\pi^2} \int_0^{1/n} |f(t)| dt \int_{1/n}^{\pi} \frac{|f(u)|}{u} du \\ &= o(n \log n). \end{aligned} \tag{12}$$

Similarly we have

$$|J_3| = o(n \log n). \tag{13}$$

Next we write J_4 as

$$\begin{aligned} J_4 &= \frac{1}{2\pi^2} \int_{1/n}^{\pi} \frac{f(t)}{t} dt \int_{1/n}^{\pi} \frac{f(u)}{u} \sum_{\nu=1}^{n} \big(\cos \nu(u-t) - \cos \nu(u+t)\big) du \\ &= \frac{1}{2\pi^2} \int_{1/n}^{\pi} \frac{f(t)}{t} dt \int_{1/n}^{\pi} \frac{f(u)}{u} \Big(\frac{\sin n(u-t)}{u-t} \\ &\qquad - \frac{\sin n(u+t)}{u+t} \Big) du + o(\log^2 n) \\ &= \frac{1}{2\pi^2} \int_{1/n}^{\pi/2} \frac{f(t)}{t} dt \int_{1/n}^{\pi} \frac{f(u)}{u} \frac{\sin n(u-t)}{u-t} du \\ &+ \frac{1}{2\pi^2} \int_{\pi/2}^{\pi} \frac{f(t)}{t} dt \int_{1/n}^{\pi} \frac{f(u)}{u} \frac{\sin n(u-t)}{u-t} du \\ &- \frac{1}{2\pi^2} \int_{1/n}^{\pi} \frac{f(t)}{t} dt \int_{1/n}^{\pi} \frac{f(u)}{u} \frac{\sin n(u+t)}{u+t} du + o(\log^2 n) \\ &= J_{4,1} + J_{4,2} - J_{4,3} + o(\log^2 n), \end{aligned} \tag{14}$$

say. Clearly we have, by (8),

$$(15)\qquad J_{4,2}=O\left(n\int_{\pi/2}^{\pi}\frac{|f(t)|}{t}dt\int_{1/n}^{\pi}\frac{|f(u)|}{u}du\right)=o(n\log n).$$

We divide $J_{4,1}$ as follows:

$$(16)\qquad J_{4,1}=\frac{1}{2\pi^2}\int_{1/n}^{\pi/2}\int_{|u-t|<1/2n}+\frac{1}{2\pi^2}\int_{1/n}^{\pi/2}\int_{|u-t|\leq 1/2n},$$

the first term of which is, in the absolut value, less than

$$n\int_{1/n}^{\pi/2}\frac{|f(t)|}{t}dt\int_{|u-t|<1/2n}\frac{|f(u)|}{u}du$$
$$=n\int_{1/n}^{\pi/2}\frac{|f(t)|}{t}dt\int_{t-1/2n}^{t+1/2n}\frac{|f(u)|}{u}du.$$

This does not exceed, using integration by parts in the inner integral,

$$n\int_{1/n}^{\pi/2}\frac{|f(t)|}{t}\left\{\frac{\Phi(t+1/2n)}{t+1/2n}-\frac{\Phi(t-1/2n)}{t-1/2n}\right\}dt$$
$$+n\int_{1/n}^{\pi/2}\frac{|f(t)|}{t}dt\int_{t-1/2n}^{t+1/2n}\frac{\Phi(u)}{u^2}du$$
$$=o(n\log n)+n\int_{1/n}^{\pi/2}\frac{|f(t)|}{t}\log\frac{t+1/2n}{t-1/2n}dt$$
$$=o(n\log n)+O\left(n\int_{1/n}^{\pi/2}\frac{|f(t)|}{t}\frac{1}{t-1/2n}\cdot\frac{1}{n}dt\right)$$
$$=o(n\log n).$$

The second integral of the right hand side of (16) is, in the absolute value, less than

$$(17)\qquad \frac{1}{2\pi^2}\int_{1/n}^{\pi/2}\frac{|f(t)|}{t}dt\int_{t+1/2n}^{\pi}\frac{|f(u)|}{u(u-t)}du$$
$$+\frac{1}{2\pi^2}\int_{3/2n}^{\pi/2}\frac{|f(t)|}{t}dt\int_{1/n}^{t-1/2n}\frac{|f(u)|}{u(u-t)}du,$$

the first term of which is, by integration by parts and (8),

$$\leqq\frac{1}{2\pi^2}\int_{1/n}^{\pi/2}\frac{|f(t)|}{t}\left[\frac{\Phi(u)}{u(u-t)}\right]_{t+1/2n}^{\pi}dt+\frac{1}{2\pi^2}\int_{1/n}^{\pi/2}\frac{|f(t)|}{t}dt\int_{t+1/2n}^{\pi}\frac{\Phi(u)(2u-t)}{u^2(u-t)^2}du$$
$$\leqq\frac{\Phi(\pi)}{2\pi^3}\int_{1/n}^{\pi/2}\frac{|f(t)|}{t(\pi-t)}dt+o\left(n\int_{1/n}^{\pi/2}\frac{|f(t)|}{t}dt\right)+o\left(\int_{1/n}^{\pi/2}\frac{|f(t)|}{t}dt\int_{t+1/2n}^{\pi}\frac{du}{(u-t)^2}\right)$$
$$=o(\log n)+o(n\log n)+o\left(n\int_{1/n}^{\pi/2}\frac{|f(t)|}{t}dt\right)$$
$$=o(n\log n).$$

Similarly we can prove that the second term of (17) is also $o(n \log n)$. Thus we get

$$J_{4,1} = o(n \log n) . \tag{18}$$

Lastly we shall estimate $J_{4,3}$.

$$\begin{aligned} |J_{4,3}| &\leqq \frac{1}{2\pi^2} \int_{1/n}^{\pi/2} \frac{|f(t)|}{t} dt \int_{1/n}^{\pi} \frac{|f(u)|}{u} \frac{du}{u+t} \\ &\leqq \frac{1}{2\pi^2} \int_{1/n}^{\pi/2} \frac{|f(t)|}{t} dt \int_{1/n}^{\pi} \frac{|f(u)|}{u^2} du \\ &= o(n \log n) , \end{aligned} \tag{19}$$

by (9).

Combining (14), (15), (18) and (19) we have

$$J_4 = o(n \log n) . \tag{20}$$

Thus by (11), (12), (13) and (20) the proof is complete.