PAPERS COMMUNICATED

7. *An Abstract Integral.*

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The object of this paper is to define abstract integrals different from Riemann and Lebesgue integrals.*)

§ *1. Riemann integrals.*[1]

1.1. Jordan fields. We consider an arbitrary set X whose points will be denoted by $x, y, \dots$ A Jordan field $\mathfrak{X}$ is a class of subsets of X whose general element will be denoted by E with the properties:

1) $0 \varepsilon \mathfrak{X}$; if $E \varepsilon \mathfrak{X}$, then $CE = X - E \varepsilon \mathfrak{X}$;

2) if $E_1, E_2 \varepsilon \mathfrak{X}$, then $E_1E_2 \varepsilon \mathfrak{X}$, $E_1 + E_2 \varepsilon \mathfrak{X}$;

and a numerical function on $\mathfrak{X}$ (Jordan measure) which will be denoted by $|E| = vE$, has properties:

3) $0 \leqq vE \leqq 1$, $|0| = 0$, $|X| = 1$;

4) if $E_1E_2 = 0$, then $|E_1 + E_2| = |E_1| + |E_2|$.

1.2. Partition. A partition (of X) which will be denoted by $\delta = (E_\nu)$ is a representation of X as a finite sum of mutually exclusive elements $E_\nu \varepsilon \mathfrak{X} (\nu = 1, 2, \dots, n)$; the set E_ν will be called an element of δ.

If $\delta = (E_\nu)$ and $\delta' = (E'_\nu)$ are two partitions, then we put $\delta < \delta'$, provided that each element E'_ν of δ' is contained in some element E_μ of δ and $\delta \neq \delta'$. Partition $\delta = (E_\nu)$ is said to be regular when $|E_\nu| > 0$ for all ν.

1.3. Riemann integrals. Let $f(x)$ be a bounded real function on X. Corresponding to any partition $\delta = (E_\nu)$ we form the "Riemann sums":

$$\overline{M}(\delta) = \sum_\nu \underset{x \varepsilon E_\nu}{\text{l. u. b.}} f(x) \cdot |E_\nu|, \quad \underline{M}(\delta) = \sum_\nu \underset{x \varepsilon E_\nu}{\text{g. l. b.}} f(x) \cdot |E_\nu|$$

and their limits

$$\overline{M} = \lim_\delta \overline{M}(\delta), \quad \underline{M} = \lim_\delta \underline{M}(\delta),$$

where the limits are taken in the sense that, for any $\varepsilon > 0$, there exists a partition δ_ε such that $|\overline{M} - \overline{M}(\delta)| < \varepsilon$ and $|\underline{M} - \underline{M}(\delta)| < \varepsilon$ for all $\delta > \delta_\varepsilon$.

Obviously $\overline{M} \geqq \underline{M}$. The function $f(x)$ is called (R)-integrable if $\overline{M} = \underline{M}$. The common value is called (R)-integral and is denoted by

$$(R)\int_X f(x)\, dv.$$

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*) Cf. S. Izumi, Jap. Journ. of Math., **13** (1936), where an abstract integral different from (R)- and (L)-integrals is given. Integrals here given seem as a bridge between the above two.

1) Cf. S. Bochner, Annals of Math., **40** (1939).

§ 2. *Lebesgue integrals.*

2.1. Lebesgue fields. A Lebesgue field $\overline{\mathfrak{X}}$ is a Jordan field whose general element will be denoted by $\overline{E}$ and the corresponding numerical function (Lebesgue measure) by $m\overline{E}=m(\overline{E})$ instead of $v\overline{E}$, which has the following properties:

5) if $\overline{E}_\nu \varepsilon \overline{\mathfrak{X}} (\nu=1, 2, \dots)$, then $\overline{E}_1+\overline{E}_2+\cdots \varepsilon \overline{\mathfrak{X}}$;

6) if $\overline{E}_\nu$ converges monotonously to $\overline{E}$, then $m\overline{E}_\nu$ converges to $m\overline{E}$.

7) any subset of a set in $\overline{\mathfrak{X}}$ of Lebesgue measure 0 is again an element of the field $\overline{\mathfrak{X}}$.

It is proved by Jessen[1] that a Jordan field $\mathfrak{X}$ can be extended to a Lebesgue field provided that the assumption

$$E_1 \supset E_2 \supset E_3 \supset \cdots\cdots \text{ and } \lim E_n = 0$$

imply

$$\lim_{n\to\infty} |E_n| = 0 .$$

Bochner[2] has proved that a "generated" Jordan field of the module C can be extended to a Lebesgue field if and only if for any sequence $\{f_n(x)\} \subset C$ the assumptions

$$\lim_{n\to\infty} f_n(x) = 0 , \quad |f_n(x)| \leqq K \qquad (n=1, 2, \dots)$$

imply

$$\lim_{n\to\infty} (R)\int_X f_n(x)dv = 0 .$$

2.2. Lebesgue integrals. Let $\overline{\mathfrak{X}}$ be a given Lebesgue field. The function $f(x)$, defined in X, is called ($\overline{\mathfrak{X}}$)-measurable or simply measurable if the set $\underset{x}{E}\big(f(x) \geqq a\big)$ belongs to $\overline{\mathfrak{X}}$ for all real a.

Let $\delta=(I_\nu)$ be an "enumerable partition" of $(-\infty, \infty)$ and let us put $I_\nu=(a_\nu, a_{\nu+1})$ and $E_\nu=\underset{x}{E}\big(a_{\nu+1} > f(x) \geqq a_\nu\big)$. If $f(x)$ is measurable and the series $\sum f(a_\nu)\, m(E_\nu)$ is absolutely convergent and the limit

$$\lim_{\delta} \sum f(a_\nu)\, m(E_\nu)$$

exists, then $f(x)$ is called (L)-integrable and the (L)-integral is defined by the limiting value. It will be denoted by

$$(L)\int_X f(x)\, dm .$$

§ 3. *Double field integrals.*[3]

3.1. Double fields. Suppose that a Jordan field $\mathfrak{X}$ and a Lebesgue field $\overline{\mathfrak{X}}$ are defined in X, where general elements of $\mathfrak{X}$ and $\overline{\mathfrak{X}}$ are denoted

1) Jessen, Matem. Tidsk., 1934.

2) S. Bochner, loc. cit.

3) Cf. A. Denjoy, Comptes Rendus, **169** (1919); S. Kempisty, Ann. de la soc. polonaise, 1929 and 1930.

by E and $\bar{E}$ respectively, with the following relations:

8) if $E\varepsilon\mathfrak{X}$, $\bar{E}\varepsilon\bar{\mathfrak{X}}$, then $E\cdot\bar{E}\varepsilon\bar{\mathfrak{X}}$;

such field will be called double field or $(\mathfrak{X}, \bar{\mathfrak{X}})$-field.

If $E\varepsilon\mathfrak{X}$, $\bar{E}\varepsilon\bar{\mathfrak{X}}$, then $C\bar{E}\varepsilon\bar{\mathfrak{X}}$ and $E=\bar{E}\cdot E+C\bar{E}\cdot E\varepsilon\bar{\mathfrak{X}}$. Hence all sets in $\mathfrak{X}$ are contained in $\bar{\mathfrak{X}}$. But vE need not be the constant multiple of mE, for otherwise Jordan field $\mathfrak{X}$ must be extended to the Lebesgue field.

3.2. Let $f(x)$ be finite and be $(\bar{\mathfrak{X}})$-measurable in X, and $|E|>0$. By $\Phi'=\Phi'(E, \lambda)$ we denote the least upper bound of y' such that

$$m\Big(E\cdot \underset{x}{E}\big(f(x)<y'\big)\Big)\leqq\lambda\,|E|,$$

where λ is positive and taken sufficiently small. And we denote by $\Phi=\Phi(E, \lambda)$ the greatest lower bound of y such that

$$m\Big(E\cdot \underset{x}{E}\big(f(x)<y\big)\Big)=m\Big(E\cdot \underset{x}{E}\big(f(x)<\Phi'\big)\Big),$$

if finite; otherwise by $\Phi=\Phi'$. Therefore

$$m\Big(E\cdot \underset{x}{E}\big(f(x)<\Phi\big)\Big)\leqq\lambda\,|E|.$$

Similarly we define $\Psi'=\Psi'(E, \lambda)$ as the greatest lower bound of z' such that

$$m\Big(E\cdot \underset{x}{E}\big(f(x)>z'\big)\Big)\leqq\mu\,|E|,$$

μ being taken sufficiently large, and define $\Psi=\Psi(E, \mu)$ as the least upper bound of z such that

$$m\Big(E\cdot \underset{x}{E}\big(f(x)>z\big)\Big)=m\Big(E\cdot \underset{x}{E}\big(f(x)>\Psi'\big)\Big),$$

if finite, otherwise let $\Psi=\Psi'$. Thus

$$m\Big(E\cdot \underset{x}{E}\big(f(x)>\Psi\big)\Big)\leqq\mu\,|E|.$$

$\Phi(E, \lambda)$ tends to "essential maximum" of $f(x)$ in E as $\lambda\to\infty$, and $\Psi(E, \mu)$ tends to "essential minimum" of $f(x)$ in E as $\mu\to 0$.

3.3. Double field integrals. Let us suppose that the $(\mathfrak{X}, \bar{\mathfrak{X}})$-field is given in X and $f(x)$ is $(\bar{\mathfrak{X}})$-measurable. Corresponding to any regular partition $\delta=(E_\nu)$ $(E_\nu\varepsilon\mathfrak{X})$ we form the Riemann sums

$$\bar{M}_\lambda(\delta)=\sum_\nu \Phi(E_\nu, \lambda)\cdot|E_\nu|,\quad \underline{M}_\mu(\delta)=\sum_\nu \Psi(E_\nu, \mu)\cdot|E_\nu|,$$

and their limits

$$\bar{\bar{M}}=\lim_{\lambda\to\infty}\overline{\lim_{\delta}}\,\bar{M}_\lambda(\delta),\quad \underline{\underline{M}}=\lim_{\mu\to 0}\underline{\lim_{\delta}}\,\underline{M}_\mu(\delta).$$

We have easily $\bar{\bar{M}}\geqq\underline{\underline{M}}$. The function $f(x)$ is called $(\mathfrak{X}, \bar{\mathfrak{X}})$-integrable if $\bar{\bar{M}}=\underline{\underline{M}}$. The common value will be denoted by

$$(\mathfrak{X}, \bar{\mathfrak{X}})\int_X f(x)\,dv.$$

3.4. When $\bar{\mathfrak{X}}$ is replaced by a Jordan field $\mathfrak{Y}$, we can define the $(\mathfrak{X}, \mathfrak{Y})$-integral by the above method.

4. Relation between above integrals.

4.1. (R)-integrals and $(\mathfrak{X}, \mathfrak{Y})$-integrals. We will consider a general set X and a double field $(\mathfrak{X}, \mathfrak{Y})$ in X. If $\mathfrak{Y}$ is a Banach field,[1] that is, a Jordan field to which all subsets of X belong, then the $(\mathfrak{X}, \mathfrak{Y})$-integral becomes a "generalization" of (R)-integral of the field $\mathfrak{Y}$. For bounded functions their $(\mathfrak{X}, \mathfrak{Y})$-integrals are equal to the (R)-integrals. Let $\mathfrak{X}$ be a generated Jordan field of module C. Then for any $f(x) \varepsilon C$, $\underset{x}{E}\big(f(x) \geqq a\big) \varepsilon \mathfrak{X}$ except a in a null set H. We define $\Phi' = \Phi'(E, \lambda)$ in § 3.2 as the least upper bound of $y' \bar{\varepsilon} H$ such that

$$\left| E \cdot \underset{x}{E}\big(f(x) < y'\big) \right| \leqq \lambda \, | E |$$

for $E \varepsilon \mathfrak{X}$. Defining Ψ' similarly, we get an integral slightly different from the $(\mathfrak{X}, \bar{\mathfrak{X}})$-integral. We denote it by $(\mathfrak{X})$-integral. For $f(x) \varepsilon C$ the $(\mathfrak{X})$-integral of $f(x)$ coincides with its (R)-integral of the generated field $\mathfrak{X}$; but $(\mathfrak{X}, \bar{\mathfrak{X}})$-integral is not so in general.[2]

4.2. (R)- and (L)-integrals and $(\mathfrak{X}, \bar{\mathfrak{X}})$-integrals. If X is a finite interval in the one dimensional space, and $\mathfrak{X}$ and $\bar{\mathfrak{X}}$ are ordinary Jordan and Lebesgue fields, then $(\mathfrak{X}, \bar{\mathfrak{X}})$-integral becomes the (A)-integral due to Denjoy.[3] It is known that (A)-integral is equivalent to the Lebesgue integral.

Let X be a general set. If $\mathfrak{X}$ is a generated Jordan field of a module C, then $f(x)$ in C is $(\bar{\mathfrak{X}})$-measurable. If $\bar{\mathfrak{X}}$ is a Lebesgue extension of $\mathfrak{X}$, then $(\mathfrak{X}, \bar{\mathfrak{X}})$-integral of $f(x)$ coincides with the (R)-integral of the field $\mathfrak{X}$.

$(\mathfrak{X}, \bar{\mathfrak{X}})$-integral is not necessarily a generalization of an (L)-integral and vice versa. For, let $\mathfrak{X}$ be a generated Jordan field of a module C and let us suppose that $(\mathfrak{X}, \bar{\mathfrak{X}})$-integral is an (L)-integral. Then by the Lebesgue's convergence theorem, we see that

$$f_n(x) \varepsilon C, \quad \lim_{n\to\infty} f_n(x) = 0, \quad |f_n(x)| \leqq K \qquad (n = 1, 2, \dots)$$

imply

$$\lim_{n\to\infty} (\mathfrak{X}, \bar{\mathfrak{X}}) \int_X f_n(x) \, dv = 0 .$$

Since the left hand side integral may become (R)-integral for all $f(x) \varepsilon C$, $\mathfrak{X}$ can be extended to a Lebesgue field by the Bochner's theorem above quoted.

§ 5. Properties of double field integrals.

5.1. We will omit elementary properties of the integrals. But we state two theorems which was proved by Bochner for (R)-integrals.[2]

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1) S. Banach, Théorie des Operation linéaire.
2) S. Bochner, loc. cit.
3) A. Denjoy, Comptes Rendus, **193** (1931).

Let $\mathfrak{F}_p$ be the space of $(\bar{\mathfrak{X}})$-measurable functions such that for each $f(x)\varepsilon\mathfrak{F}_p$ the integral $(\mathfrak{X}, \bar{\mathfrak{X}})\int_X |f(x)|^p dv$ exists and the norm is defined by

$$\|f(x)\|_p=\left\{(\mathfrak{X}, \bar{\mathfrak{X}})\int_X |f(x)|^p dv\right\}^{1/p} \qquad (p\geqq 1).$$

$\mathfrak{F}_p$ is a complete normed space in the Banach sense. We can conclude that the step-functions are dense in $\mathfrak{F}_p$. If $\mathfrak{F}'_p$ is defined by $(\mathfrak{X}, \mathfrak{Y})$-integral, then the step-functions are also dense in $\mathfrak{F}'_p$.

5.2. Let $p>1$. By V_p we denote the space of set-functions defined in X such that for each $F(E)\varepsilon V_p$ there exists a limit $\lim_\delta A(\delta)$, where $\delta=(E_\nu)$ is any regular partition and

$$A(\delta)=\left(\sum_\nu |F(E_\nu)|^p |E|^{1-p}\right)^{1/p}$$

and the norm is defined by

$$\|F\|_p=\lim_\delta A(\delta).$$

Then V_p becomes a complete normed space. We can prove that V_p is equal to the closure of $\mathfrak{F}_p$ and is also equal to the closure of $\mathfrak{F}'_p$.

When $X=(0, 1)$, and $\mathfrak{X}$ and $\bar{\mathfrak{X}}$ are ordinary Riemann and Lebesgue fields, $(\mathfrak{X}, \bar{\mathfrak{X}})$-integral is equal to (A)-integral and then to (L)-integral, and $\mathfrak{F}_p$ is closed. Thus the above result becomes the Young's theorem. For suitable $\mathfrak{X}$ and $\mathfrak{Y}$, $(\mathfrak{X}, \mathfrak{Y})$-integral becomes (R)-integral as already shown. Then $cl(\mathfrak{F}'_p)=R_p$ by the notation of Bochner. Thus the above result contains the Bochner's theorem in this case.

If $p=1$, we get a similar theorem taking AC instead of V_1.