21. An Abstract Integral, II.

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Introduction. The object of this paper is to define Riemann and Lebesgue integrals of functions whose values belong to a very general space. For this purpose, we will define new definitions of Riemann and Lebesgue integrals of real-valued functions of a real variable. More generally, the definitions of Riemann-Stieltjes and Lebesgue-Stieltjes integrals are given in §1 and §2, which are free from the notions of partition, least upper bound and greatest lower bound. These definitions make us easy to generalize these two integrals into abstract spaces. This is given in §3 and §4. The properties and applications of these integrals are left to the next occasion.

1. Let $f(x)$ and $z(x)$ be real-valued functions defined in the interval (a, b) and $z(x)$ be non-negative, montone and $z(b)-z(a)=1$. The sequence $(x_{n\nu})(n=1, 2, 3, \dots;\ \nu=1, 2, \dots, n)$ is said to satisfy the $(C, 1)$ condition, provided that:

1.1.° $a \leqq x_{n\nu} \leqq b$.

1.2.° For any subinterval (α, β) of (a, b) such that $z(x)$ is continuous at $x=\alpha$ and $x=\beta$,

$$\lim_{n\to\infty} (M_n/n) = z(\beta)-z(\alpha)\,,$$

where M_n denotes the number of $x_{n\nu}$ in (α, β) for fixed n.

If the limit

$$\lim_{n\to\infty} \frac{1}{n}\sum_{\nu=1}^{n} f(x_{n\nu}) \tag{1}$$

exists for any $(x_{n\nu})$ satisfying the $(C, 1)$ condition and the limiting value is independent of the choice of $(x_{n\nu})$, then we denote it by

$$(\mathfrak{R})\int_a^b f(x)dz(x)$$

and we say that $f(x)$ is $(\mathfrak{R})$-integrable.

If $f(x)$ is continuous, then it is easy to verify that $f(x)$ is $(\mathfrak{R})$-integrable. It is known that, if $f(x)$ is continuous, (1) is equal to the Riemann-Stieltjes integral (or simply (RS)-integral) of $f(x)$ by the determinate function $z(x)$.[1] Further we can prove that $(\mathfrak{R})$-integral is equivalent to the (RS)-integral.[2]

2. Let $f(x)$ be a real-valued measurable function in (a, b) and $z(e)$

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1) J. Schoenberg, Math. Zeits., **28** (1928).
2) Cf. I. J. Ridder, Prace Mat.-Fys., 1936.

be a real-valued set-function of measurable set e contained in $E=(a, b)$ such that

2.1°. $z(e)$ is non-negative and monotone, that is, $0 \leqq z(e) \leqq z(e')$ if $e \subseteqq e'$.

2.2°. $z(e)$ is completely additive, that is, if $e=\sum_{n=1}^{\infty} e_n$, $e_i \cdot e_j = 0$ $(i \neq j)$, then $z(e)=\sum_{n=1}^{\infty} z(e_n)$.

2.3°. $z(E)=1$.

The sequence $(x_{n\nu})$ is said to satisfy the $(C, 2)$ condition provided that:

2.4°. $a \leqq x_{n\nu} \leqq b$.

2.5°. For any set $e=\underset{x}{E}(\alpha \leqq f(x) < \beta)$, α and β being real numbers,

$$\lim_{n\to\infty} (N_n/n) = z(e),$$

where N_n denotes the number of $x_{n\nu}$ in e, n being fixed.

If the limit

$$\lim_{n\to\infty} \frac{1}{n} \sum_{\nu=1}^{n} f(x_{n\nu}) \tag{2}$$

exists for any $(x_{n\nu})$ satisfying the $(C, 2)$ condition and the limiting value is independent of the choice of $(x_{n\nu})$, then we say that $f(x)$ is $(\mathfrak{L})$-integrable and (2) is denoted by

$$(\mathfrak{L})\int_E f(x) z(dE).$$

We can prove that $(\mathfrak{L})$-integral is equivalent to the Lebesgue-Radon integral.[3)]

If we replace the condition 2.2° by the following

2.2′. $z(e)$ is finitely additive, that is, if $e=\sum_{n=1}^{m} e_n$ and $e_i \cdot e_j = 0$ $(i \neq j)$, then $z(e)=\sum_{n=1}^{m} z(e_n)$,

then we get another integral which is equivalent to another Lebesgue-Radon integral.[3)]

3. Let E and E_1 be abstract spaces such that the Riemann field $\mathfrak{X}$ is defined in E such as $\mu E=1$, μ being Jordan measure,[4)] and E_1 is a linear (or vectorial) space where the notion of limit satisfying ordinary axioms is defined. Let $f(x)$ be a function whose domain is E and range is contained in E_1. The sequence $(x_{n\nu})$ is said to satisfy the $(C, 1')$ condition provided that:

3.1°. $x_{n\nu} \in E (n=1, 2, 3, \dots;\ \nu=1, 2, \dots, n)$

3.2°. For any set $e \in \mathfrak{X}$,

$$\lim_{n\to\infty} (M'_n/n) = \mu e$$

3) Cf. J. Hildebrandt, Trans. Am. Math. Soc., **41** (1937), and Fichtenholz-Kantrovitch, Studia Math., **6** (1937).

4) Cf. S. Bochner, Annals of Math., **42** (1939), and S. Izumi, this Proceedings, 1940.

where M'_n denotes the number of $x_{n\nu}$ in e, n being fixed.

If the limit

$$\lim_{n\to\infty}\frac{1}{n}\sum_{\nu=1}^{n}f(x_{n\nu}) \tag{3}$$

exists for any $(x_{n\nu})$ satisfying the $(C, 1')$ condition and the limiting value is independent of the choice of $(x_{n\nu})$, then we say that $f(x)$ is $(\mathfrak{R}')$-integrable and the limiting value is denoted by

$$(\mathfrak{R}')\int_E f(x)d\mu .$$

We can prove that uniformly continuous functions and more generally the functions uniformly continuous except a μ-null set are $(\mathfrak{R}')$-integrable.

When E_1 is a Banach space, we can take the limit in (3) as that in the norm sense and as that in the weak topology. Thus we get two kinds of $(\mathfrak{R}')$-integrals, that is, the "strong $(\mathfrak{R}')$-integral" and "weak $(\mathfrak{R}')$-integral".

4. Let $f(x)$ be a function whose domain is E and whose range is E_1. Suppose that the Lebesgue field $\bar{\mathfrak{X}}$ is defined in E with $mE=1$, m being Lebesgue measure, and E_1 is a linear metric space.[4]

Let $f(x)$ be a $(\bar{\mathfrak{X}})$-measurable function, that is, $\underset{x}{E}\big(f(x)\varepsilon K\big)\varepsilon\bar{\mathfrak{X}}$ for any sphere K in E_1. $(x_{n\nu})$ is said to satisfy the $(C, 2')$ condition, provided that:

4.1°. $x_{n\nu}\varepsilon E$ $(n=1, 2, 3, \dots ;\ \nu=1, 2, \dots, n)$,

4.2. For any set $\bar{e}=\underset{x}{E}\big(f(x)\varepsilon K\big)$, $\mathfrak{S}$ being a sphere in E_1,

$$\lim_{n\to\infty}(N'_n/n)=m\bar{e} ,$$

where N'_n denotes the number of $x_{n\nu}$ in e, n being fixed.

If the limit

$$\lim_{n\to\infty}\frac{1}{n}\sum_{\nu=1}^{n}f(x_{n\nu}) \tag{4}$$

exists for any $(x_{n\nu})$ satisfying the $(C, 2')$ condition and the limiting value is independent of the choice of $(x_{n\nu})$, then we say that $f(x)$ is $(\mathfrak{L}')$-integrable and (4) is denoted by

$$(\mathfrak{L}')\int_E f(x)dm.$$

We can prove that bounded $\bar{\mathfrak{X}}$-measurable functions are $(\mathfrak{L}')$-integrable and that the $(\mathfrak{L}')$-integral is absolutely convergent.

When E_1 is a Banach space, we can define the "strong" and "weak" integrals as in the preceding section.[5]

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5) In this direction there are papers of G. Birkhoff, Dunford, Pettis, I. Gelfand and other writers.