19. *On Some Properties of Umbilical Points of Hypersurfaces.*

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(1) Let us consider in an $n+1$-dimensional Riemannian space V_{n+1} a hypersurface V_n denoted by

$$x^\lambda = x^\lambda(x^i) \qquad \begin{cases} \lambda, \mu, \nu, \dots = 1, 2, \dots, n+1 \\ i, j, k, \dots = 1, 2, \dots, n. \end{cases}$$

Then we get the following relations:

$$N_\lambda \partial_i x^\lambda = 0, \qquad g_{\lambda\mu} N^\lambda N^\mu = 1,$$

$$H_{jk}^{\cdot\cdot\lambda} = \partial_k \partial_j x^\lambda + \{^{\lambda}_{\mu\nu}\} \partial_j x^\mu \partial_k x^\nu - \{^{i}_{jk}\} \partial_i x^\lambda = -N_{jk} N^\lambda,$$

where N^λ is the unit vector field normal to V_n and $\{^{\lambda}_{\mu\nu}\}$ and $\{^{i}_{jk}\}$ are the Christoffel symbols constructed from the fundamental tensors $g_{\mu\nu}$ and g_{jk} of V_{n+1} and V_n respectively. From the second fundamental tensor N_{jk} we can construct the quantity

$$M_{jk} = N_{jk} - \frac{1}{n} g_{jk} N_{lm} g^{lm} \tag{1.1}$$

which is only multiplied by ρ under the transformation $g_{\mu\nu} \to \rho^2 g_{\mu\nu}$.

A line of curvature is a curve $x^i(t)$ which satisfies the equations

$$M^i_{\ j} \dot{x}^j = \alpha \dot{x}^i. \tag{1.2}$$

When we differentiate (1.2) with respect to t we get

$$M^i_{\ j} a^j + M^i_{\ jk} \dot{x}^j \dot{x}^k = \alpha a^i + \dot{\alpha} \dot{x}^i, \tag{1.3}$$

$$M^i_{\ j} b^j + 2M^i_{\ jk} a^j \dot{x}^k + M^i_{\ jk} \dot{x}^j a^k + M^i_{\ jkl} \dot{x}^j \dot{x}^k \dot{x}^l = \alpha b^i + 2\dot{\alpha} a^i + \ddot{\alpha} \dot{x}^i, \tag{1.4}$$

$$\begin{aligned} &M^i_{\ j} c^j + 3M^i_{\ jk} b^j \dot{x}^k + M^i_{\ jk} \dot{x}^j b^k + 3M^i_{\ jk} a^j a^k + 3M^i_{\ jkl} a^j \dot{x}^k \dot{x}^l \\ &+ 2M^i_{\ jkl} \dot{x}^j a^k \dot{x}^l + M^i_{\ jkl} \dot{x}^j \dot{x}^k a^l + M^i_{\ jklm} \dot{x}^j \dot{x}^k \dot{x}^l \dot{x}^m \\ &= \alpha c^i + 3\dot{\alpha} b^i + 3\ddot{\alpha} a^i + \dddot{\alpha} \dot{x}^i, \end{aligned} \tag{1.5}$$

where

$$\begin{aligned} a^i &= \ddot{x}^i + \{^{i}_{jk}\} \dot{x}^j \dot{x}^k \\ b^i &= \dot{a}^i + \{^{i}_{jk}\} a^j \dot{x}^k \\ c^i &= \dot{b}^i + \{^{i}_{jk}\} b^j \dot{x}^k \end{aligned} \tag{1.6}$$

and $M^i_{\ jk}$, $M^i_{\ jkl}$ etc. are the covariant derivatives of $M^i_{\ j}$ with respect to g_{jk}.

We call a point on the V_n a perfectly umbilical point when there is a line of curvature passing through the point in each direction.

Let us consider the equations satisfied by $M^i{}_j$ and its derivatives at a perfectly umbilical point.

As $\dot{x}^i$ are arbitrary at such a point, we get at first from (1.2)

$$M^i{}_j = a\delta^i{}_j\,, \qquad M_{jk} = a g_{jk}\,.$$

hence

(1.7) $$M_{jk}=0\,, \qquad a=0\,,$$

because of (1.1) or

(1.8) $$M^i{}_i = M_{jk}g^{jk}=0\,.$$

Then (1.3) becomes

$$M^i{}_{jk}\dot{x}^j\dot{x}^k = \dot{a}\dot{x}^i\,, \qquad \dot{a}=\frac{M_{ijk}\dot{x}^i\dot{x}^j\dot{x}^k}{g_{lm}\dot{x}^l\dot{x}^m}$$

and as $\dot{x}^i$ are arbitrary we obtain

$$M_{i(ab}g_{cd)} = g_{i(a}M_{bcd)}$$

or

(1.9) $$\begin{aligned}(M_{iab}+M_{iba})g_{cd}&+(M_{iac}+M_{ica})g_{bd}+(M_{iad}+M_{ida})g_{bc}\\ &+(M_{icd}+M_{idc})g_{ab}+(M_{ibd}+M_{idb})g_{ac}+(M_{ibc}+M_{icb})g_{ad}\\ &=g_{ia}(M_{bcd}+M_{cdb}+M_{dbc})+g_{ib}(M_{acd}+M_{cda}+M_{dac})\\ &+g_{ic}(M_{abd}+M_{bda}+M_{dab})+g_{id}(M_{abc}+M_{bca}+M_{cab})\,.\end{aligned}$$

Multiplying by g^{id} we get

(1.10) $$(n+1)(M_{abc}+M_{bca}+M_{cab})+M_a{}^l{}_l g_{bc}+M_b{}^l{}_l g_{ac}+M_c{}^l{}_l g_{ab}$$

on account of (1.8). Multiplying (1.10) by g_{bc} we get

(1.11) $$M_a{}^l{}_l=0\,.$$

Then multiplying (1.9) by g^{cd} we get

$$(n+2)(M_{iab}+M_{iba})-2M_{abi}=0\,.$$

From these equations, (1.10), and (1.11) we obtain

(1.12) $$M_{ijk}=0\,.$$

(2) Now Codazzi's equations in the conformal geometry of Riemannian spaces are the following[1]:

(2.1) $$M_{ijk}-M_{ikj}+\frac{1}{n-1}(M^a{}_{ka}g_{ij}-M^a{}_{ja}g_{ik})-N_\lambda C^\lambda{}_{\omega\mu\nu}\partial_i x^\omega \partial_j x^\mu \partial_k x^\nu=0\,.$$

As at a perfectly umbilical point (2.1) becomes

$$N_\lambda C^\lambda{}_{\omega\mu\nu}\partial_i x^\omega \partial_j x^\mu \partial_k x^\nu=0$$

we can state the following theorem:

Theorem: *The Weyl conformal curvature of a Riemannian space vanishes at a point if at this point in each direction there is a hypersurface with the point as a perfectly umbilical point.*

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1) K. Yano: Proc. **15** (1939), 340.

Therefore we consider in the next section the case where V_{n+1} is conformally flat.

(3) When V_{n+1} is conformally flat Codazzi's equations become

$$M_{ijk}-M_{ikj}+\frac{1}{n-1}(M^{a}{}_{ka}g_{ij}-M^{a}{}_{ja}g_{ik})=0\,, \tag{3.1}$$

which give on differentiation

$$M_{ijkl}-M_{ikjl}+\frac{1}{n-1}(L_{kl}g_{ij}-L_{jl}g_{ik})=0 \tag{3.2}$$

where

$$L_{jk}=M^{a}{}_{jak}\,. \tag{3.3}$$

At a perfectly umbilical point (1.4) becomes

$$M^{i}{}_{jkl}\dot{x}^{j}\dot{x}^{k}\dot{x}^{l}=\ddot{a}\dot{x}^{i}$$

because of (1.12) and as $\dot{x}^{i}$ are arbitrary we get

$$M_{i(abc}g_{de)}=g_{i(a}M_{bcde)}\,. \tag{3.4}$$

As M_{ijkl} is symmetric in k and l because of Ricci identity and (1.7) we get from (3.4)

$$\begin{aligned} &g_{ab}(L_{cd}+L_{dc})+g_{ac}(L_{bd}+L_{db})+g_{ad}(L_{bc}+L_{cb})\\ &+g_{cd}(L_{ab}+L_{ba})+g_{bd}(L_{ac}+L_{ca})+g_{bc}(L_{ad}+L_{da})\\ &=(n+2)\,(M_{abcd}+M_{acbd}+M_{adbc}+M_{cdab}+M_{bdac}+M_{bcad})\,, \end{aligned} \tag{3.5}$$

hence

$$2Lg_{ab}+(n+4)\,(L_{ab}+L_{ba})=(n+2)\,\{T_{ab}+2(L_{ab}+L_{ba})\}\,,$$

$$(n+2)\,T_{ab}=-n(L_{ab}+L_{ba})+2Lg_{ab}\,, \tag{3.6}$$

where

$$T_{ij}=M_{ijkl}g^{kl}\,. \tag{3.7}$$

On the other hand we obtain from (3.2)

$$T_{ij}-L_{ij}+\frac{1}{n-1}(Lg_{ij}-L_{ji})=0\,,$$

$$T_{(ij)}-L_{(ij)}+\frac{1}{n-1}(Lg_{(ij)}-L_{(ji)})=0\,,$$

that is,

$$T_{ij}=T_{(ij)}=\frac{1}{2(n-1)}\{n(L_{ij}+L_{ji})-2Lg_{ij}\}\,.$$

These equations and (3.6) give

$$T_{ab}=0\,, \tag{3.8}$$

$$L_{ab}+L_{ba}=\frac{2}{n}Lg_{ab}\,. \tag{3.9}$$

Multiplying (3.4) by g^{de} we get

$$(n+6)(M_{iabc}+M_{ibca}+M_{icab})+2(L_{ia}g_{bc}+L_{ib}g_{ca}+L_{ic}g_{ab})$$
$$=2\{(L_{ab}+L_{ba})g_{ic}+(L_{bc}+L_{cb})g_{ia}+(L_{ca}+L_{ac})g_{ib}\}$$
$$+2(M_{iabc}+M_{ibca}+M_{icab}+M_{bcia}+M_{caib}+M_{abic}),$$

because of (3.8). Multiplying by g^{bc} again we get

$$(n+4)L_{ia}=2(L_{ia}+L_{ai})+Lg_{ia},$$

hence

(3.10) $$L_{ij}=\frac{1}{n}Lg_{ij}$$

because of (3.9).

Therefore (3.5) becomes

$$M_{abcd}+M_{acbd}+M_{adbc}+M_{cdab}+M_{bdac}+M_{bcad}$$
$$=\frac{4L}{n(n+2)}(g_{ab}g_{cd}+g_{ac}g_{bd}+g_{ad}g_{bc})$$

and as the left side becomes

$$6M_{abcd}+(M_{acbd}-M_{abcd})+(M_{adbc}-M_{abdc})+(M_{cdab}-M_{cadb})$$
$$+(M_{acbd}-M_{abcd})+(M_{bdac}-M_{badc})+(M_{bcad}-M_{bacd})$$

we obtain, by using (3.2) and (3.10),

$$6M_{abcd}-\frac{L}{n(n-1)}(2g_{ac}g_{bd}+2g_{ad}g_{bc}-4g_{ab}g_{cd})$$
$$=\frac{4L}{n(n+2)}(g_{ab}g_{cd}+g_{ac}g_{bd}+g_{ad}g_{cb}),$$

that is,

(3.11) $$M_{abcd}=\frac{L}{(n-1)n(n+2)}\{-2g_{ab}g_{cd}+n(g_{ac}g_{bd}+g_{ad}g_{bc})\}.$$

(3.2) and (3.4) are satisfied by (3.11).

At the point where (1.7), (1.12), and (3.11) are satisfied (1.5) takes the form

$$La^i=\frac{(n-1)(n+2)}{3}\frac{M^i{}_{jklm}\dot{x}^j\dot{x}^k\dot{x}^l\dot{x}^m}{g_{ab}\dot{x}^a\dot{x}^b}+\lambda\dot{x}^i$$

and from the equations obtained by differentiating (1.5) successively we get the expressions of Lb^i, Lc^i, etc. in terms of $\dot{x}^i$, so that we can solve the differential equations of the lines of curvature passing through the point with an arbitrary initial values of $\dot{x}^i$, if $L\neq 0$. Hence we obtain the following theorem:

Theorem: *At a perfectly umbilical point* (1.7), (1.12), *and* (3.11) *are satisfied. Besides, when* $L\neq 0$ *these relations are the sufficient conditions for a perfectly umbilical point.*

Especially when V_{n+1} is flat (3.11) becomes

(3.12) $$N_{ijkl}=K(g_{ij}g_{kl}+g_{ik}g_{jl}+g_{il}g_{jk}).$$