31. On the Theory of Almost Periodic Functions in a Group.

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The theory of almost periodic functions (a. p. f.) in a group, due originally to J. von Neumann,[1] has been simplified by W. Maak.[2] The last author starts from a modified definition of a. p. f., and obtains a shorter proof of the existence of the mean value. His proof necessitates, however, a certain combinatorial lemma, which is indeed very interesting in itself, but somewhat alien to the theory of a. p. f. We propose here another way of founding this theory, which seems to us also simple and natural.

1. We begin with some general remarks on metric spaces. An abstract space $\mathfrak{R}$ with "points" $x, y, z, \dots$ is called a metric space, if there is defined a "metric," i. e. a real-valued function $\rho(x, y)$ for $x, y \in \mathfrak{R}$ satisfying the following conditions: 1) $\rho(x, y) \geqq 0$, $\rho(x, x)=0$, 2) $\rho(x, y)=\rho(y, x)$, 3) $\rho(x, y)+\rho(y, z) \geqq \rho(x, z)$. The separation axiom: "$\rho(x, y)=0$ implies $x=y$" will not be postulated here.[3] Such spaces are topological spaces, i. e, they satisfy the first three Hausdorff axioms; it is therefore clear what are to be understood under the terms such as: open or closed sets in $\mathfrak{R}$, the continuity of a mapping of $\mathfrak{R}$ in another such space $\mathfrak{R}'$ etc. One can, moreover, speak of the equi-continuity of a family of mappings and also the uniform continuity of a mapping.[4] Theorems such as the following are evidently true thereby: If f maps $\mathfrak{R}$ continuously in $\mathfrak{R}'$, and f' maps $\mathfrak{R}'$ in $\mathfrak{R}''$ in the same way, then $f''=f'f$ maps $\mathfrak{R}$ also continuously in $\mathfrak{R}''$. If, moreover, f and f' are uniformly continuous, so is also f''. (Transitivity of continuity and uniform continuity.)

We can speak also of the diameter of a set $\mathfrak{A}$ in $\mathfrak{R}$, ε-covering, ε-net, the boundedness and the totally-boundedness of $\mathfrak{A}$. If we have to do with several metrics of a fixed space $\mathfrak{R}$, we will say also that *a metric ρ is bounded or totally bounded (t. b.)*, when the entire space $\mathfrak{R}$ has this property with respect to ρ. For two metrics ρ, ρ_1 of $\mathfrak{R}$ we will write $\rho \leqq \rho_1$, if $\rho(x, y) \leqq \rho_1(x, y)$ for all $x, y \in \mathfrak{R}$. The following lemmas are all fairly obvious:

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1) J. von Neumann: Almost periodic functions in a group I. Trans. Am. math. Soc. Vol. **36** (1934).

2) W. Maak: Eine neue Definition der fast periodischen Funktionen. Abh. math. Sem. d. Hans. Universität. **11.** Bd. (1936).

3) Such $\rho(x, y)$ is often called "quasi-metric" in opposition to the usual "metric" satisfying the separation axiom. We prefer to call ρ a "metric" in the general case, and "separated metric" when it satisfies the separation axiom.

4) In this sense $\mathfrak{R}$ is a "uniform space"; cf. André Weil; Espaces à structure uniforme. Act. sc. et ind. **551** (1937). A. Weil postulates, however, the separation axiom.

Lemma 1. Totally bounded metrics are bounded. If $\rho \leqq \rho_1$, *and* ρ_1 *is t. b., then* ρ *is also t. b. Let* $\mathfrak{R}'$ *be another space and* ρ', ρ_1' *metrics of* $\mathfrak{R}'$. *Suppose* $\rho \leqq \rho_1$, $\rho_1' \leqq \rho'$. *If f is a uniformly continuous mapping of* $\mathfrak{R}$ *in* $\mathfrak{R}'$, *when these spaces are metrized with* ρ, ρ' *resp. then f is also uniformly continuous, when they are metrized with* ρ_1, ρ_1'.

Lemma 2. If $\rho_1, \dots, \rho_r$ *are t. b. metrics of* $\mathfrak{R}$, *so is also* $\rho_1 + \dots + \rho_r$.

There exist namely finite ε-coverings of $\mathfrak{R}$ corresponding to ρ_i, $i=1, \dots, r$. The superposed covering of these coverings constitutes clearly an $r\varepsilon$-covering of $\mathfrak{R}$ for the metric $\rho_1 + \dots + \rho_r$.

Lemma 3. The uniform limit of t. b. metrics is also a t. b. metric; i. e. if t. b. metrics $\rho_\nu(x, y)$ *tend to* $\rho(x, y)$ *uniformly in* x, y, *then* ρ *is also a t. b. metric.*

We will add here further the following remark: Let $\mathfrak{S}$ be a set of elements $a, b, \dots$. Suppose there exist a mapping f of $\mathfrak{S}$ in a space $\mathfrak{R}$ with a metric ρ. Then $\sigma(a, b) = \rho\big(f(a), f(b)\big)$ is clearly a metric for $\mathfrak{S}$: the "transferred metric from $\mathfrak{R}$ into $\mathfrak{S}$ by means of f." If ρ is hereby t. b., so is also σ (as a metric of $\mathfrak{S}$).

2. Now let $\mathfrak{G}$ be a group, and ρ a metric of $\mathfrak{G}$. ρ is called *left-invariant* (*l-inv.*) if $\rho(ax, ay) = \rho(x, y)$ for all $a \in \mathfrak{G}$; *right-invariant* (*r-inv.*) if $\rho(xb, yb) = \rho(x, y)$ for all $b \in \mathfrak{G}$; *invariant* (*inv.*) if it is both r-inv. and l-inv. From any metric ρ we can form a l-inv. metric ρ^l, a r-inv. ρ^r and an inv. $\rho^i = \rho^{lr} = \rho^{rl}$ in putting:

$$\rho^l(x, y) = \text{l. u. b. } \rho(ax, ay)\,, \quad \rho^r(x, y) = \text{l. u. b. } \rho(xb, yb)\,,$$
$$\rho^i(x, y) = \text{l. u. b. } \rho(axb, ayb)$$

where a, b run over the elements in $\mathfrak{G}$. This process to obtain ρ^l, ρ^r, ρ^i from ρ will be called l-, r-, and *i-operation* resp.

Lemma 4. If one of the metrics ρ^l, ρ^r *and* ρ^i *is t. b., so are also the others.*

Proof. As $\rho^l \leqq \rho^i$ and $\rho^r \leqq \rho^i$, ρ^l, ρ^r are t. b. in the same time with ρ^i according to the lemma 1. Now let ρ^r be t. b. and $a_1, \dots, a_n$ an ε-net for ρ^r. Put $\rho^r(a_i x, a_i y) = \rho_i^r(x, y)$. ρ_i^r, $i=1, \dots, n$ are t. b. by the last remark in § 1. Let $\mathfrak{U}$ be the superposed covering of ε-coverings for ρ_i^r, $i=1, \dots, n$. We will see that $\mathfrak{U}$ is a 3ε-covering for $\rho^i = \rho^{rl}$ and recognize thus ρ^i as t. b. Indeed, let a be any element of $\mathfrak{G}$ and x, y two points belonging to an element of $\mathfrak{U}$. Then we have for a certain a_i $\rho^r(ax, ay) \leqq \rho^r(ax, a_i x) + \rho^r(a_i x, a_i y) + \rho^r(a_i y, ay) = 2\rho^r(a, a_i) + \rho_i^r(x, y) \leqq 3\varepsilon$, therefore $\rho^{rl}(x, y) \leqq 3\varepsilon$. The resting part is to show in the same way.

3. Let f be a complex-valued function of the elements of a group $\mathfrak{G}$. The usual metric $|\alpha - \beta|$ of the field $\mathfrak{K}$ of the complex numbers α, $\beta, \dots$, transferred into $\mathfrak{G}$ by means of f determines a metric $|f(x) - f(y)|$ of $\mathfrak{G}$. By l-, r-, and i-operation applied on this metric, we obtain the metrics[1]:

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1) It would be more consequent to put $\rho_f(x, y) = |f(x) - f(y)|$, and write ρ_f^i for ρ_f in the text. But as we have to do in the following almost exclusively with this ρ_f^i we have preferred the simpler notation.

$$\rho_f^l(x, y) = \text{l. u. b. } |f(ax) - f(ay)|, \quad \rho_f^r(x, y) = \text{l. u. b. } |f(xb) - f(yb)|,$$

$$\rho_f(x, y) = \text{l. u. b. } |f(axb) - f(ayb)|,$$

a and b running over the elements in $\mathfrak{G}$.

Definition I. We call f an almost periodic function (a. p. f.), if ρ_f is t. b.

By the lemma 4, we can say also: *if ρ_f^l or ρ_f^r is t. b.* We can give to this definition also the following forms:

Definition I'. f is a. p., if f is uniformly continuous with respect to an inv. t. b. metric of $\mathfrak{G}$.

Definition I''. f is a. p., if the family of functions $f(axb)$ (with a, b as parameters) is uniformly (in x) equi-continuous with respect to a t. b. metric of $\mathfrak{G}$.

The equivalence proof of these definitions is almost immediate: f is namely clearly uniformly continuous with respect to the inv. metric ρ_f. Thus I $\rightarrow$ I'. If f is uniformly continuous with respect to an inv. metric ρ then the family of $f(axb)$ is obviously uniformly equi-continuous with respect to ρ. Thus I' $\rightarrow$ I''. Finally I'' $\rightarrow$ I, because, if $\rho(x, y) \leqq \delta(\varepsilon)$ implies $|f(axy) - f(ayb)| \leqq \varepsilon$, a $\delta(\varepsilon)$-net for ρ constitutes an ε-net for ρ_f.

From these definitions and the preceding lemmas follow easily the known theorems:

Theorem I. Every a. p. f. is bounded. If $f_1, \dots, f_r$ are a. p. f., and $\varphi(\xi_1, \dots, \xi_r)$ is a complex-valued function of complex variables $\xi_1, \dots, \xi_r$, which is uniformly continuous for the bounded values of these variables, then $f = \varphi(f_1, \dots, f_r)$ is also a. p. The uniform limit of a. p. f. is also a. p.

We will indicate here only the proof of the second part of the theorem. Put for simplicity $\rho_i = \rho_{f_i}$, $i = 1, \dots, r$. ρ_i being inv. and t. b., $\rho = \rho_1 + \dots + \rho_r$ is also inv. and t. b. On the other hand, f_i is also uniformly continuous with respect to ρ, as $\rho_i \leqq \rho$, and so f is also unif. continuous with respect to ρ (transitivity of unif. continuity.) Therefore f is a. p. according to the definition I'.

Theorem II. Let $x \rightarrow D(x) = (D_{ik}(x))$ $(i, k = 1, \dots, d)$ be a bounded representation of $\mathfrak{G}$. The function $D_{ik}(x)$ is a. p.

Proof. The bounded part of $\mathfrak{R}$ being t. b., the transferred metric $|D_{rs}(x) - D_{rs}(y)|$ is t. b., so also the metric $\sum_{r,s} |D_{rs}(x) - D_{rs}(y)| = \rho(x, y)$. Now we have $|D_{ik}(axb) - D_{ik}(ayb)| = |\sum_{r,s} D_{ir}(a) \left(D_{rs}(x) - D_{rs}(y)\right) D_{sk}(b)| \leqq B^2 \rho(x, y)$, where B is an upper bound of $|D_{ik}(x)|$, $x \in \mathfrak{G}$, $i, k = 1, \dots, d$. The family of $D_{ik}(axb)$ is thus uniformly equi-continuous with respect to the t. b. metric ρ.

4. In this paragraph we will consider in general the complex-valued function of the elements of $\mathfrak{G}$ and the "mean value" of such functions. We begin with the

Definition II. A constant M is called an ε-approximative mean value (ε-appr. m. v.) of a function f in $\mathfrak{G}$ if there exist $x_1, \dots, x_n \in \mathfrak{G}$, *so that* $\left|\frac{1}{n}\sum_{i=1}^{n} f(ax_i;\, b)-M\right|<\varepsilon$ *holds for all* $a, b\in\mathfrak{G}$. *If* M *is an* ε-*appr. m. v. for every* $\varepsilon(>0)$, *then* M *is called a mean value* (*m. v.*) *of* f.[1)]

Lemma 5. Let f, g *be two functions in* $\mathfrak{G}$ *and* M, M' *be resp. an* ε-*appr. m. v. of* f *and an* ε'-*appr. m. v. of* g. *Then there exist certain* $z_1, \cdots, z_N$ *so that the inequalities*

$$(1)\qquad \left|\frac{1}{N}\sum_{k=1}^{N} f(az_kb)-M\right|<\varepsilon,\quad \left|\frac{1}{N}\sum_{k=1}^{N} g(az_kb)-M'\right|<\varepsilon'$$

are satisfied simultaneously for all $a, b\in\mathfrak{G}$.

Proof (by a well-known reasoning). There exist by hypothesis $x_1, \cdots, x_n$ and $y_1, \cdots, y_m$ so that

$$\left|\frac{1}{n}\sum_{i=1}^{n} f(ax_ib)-M\right|<\varepsilon,\quad \left|\frac{1}{m}\sum_{j=1}^{m} g(ay_jb)-M'\right|<\varepsilon'$$

hold for all $a, b\in\mathfrak{G}$. Put in the first inequalities y_jb for b and take the mean; in the second ax_i for a and take the same. We see thus $z_k=x_iy_j$, $k=1, \cdots, mn$ $(N=mn)$ fullfil (1).

Corollary. If M, M' are resp. an ε-appr. m. v. and an ε'-appr. m. v. of f, then holds $|M-M'|<\varepsilon+\varepsilon'$.

This Corollary affirms the uniqueness of the m. v., if any. Furthermore, the existence proof of the m. v. is reduced to that of the ε-appr. m. v. for every ε.

5. *The existence of the m. v. of a. p. f.* Let us consider a fixed a. p. f. f in $\mathfrak{G}$. We will write ρ for ρ_f and employ the following notations: We represent by X a set of elements $x_1, \cdots, x_n$ of $\mathfrak{G}$, and denote by $\mu(X)$ the mean: $\frac{1}{n}\sum_{i=1}^{n} f(x_i)$. The "translated mean" $\frac{1}{n}\sum_{i=1}^{n} f(ax_ib)$ will be denoted by $\mu(aXb)$ and the "oscillation by translation": l. u. b. $|\mu(aXb)-\mu(a'Xb')|$, where a, b, a', b' run over the elements in $\mathfrak{G}$, by Osc X. Our concern is to show that Osc X can be made $<\varepsilon$ by taking X appropriately. We will give now a procedure to diminish this Osc. in changing X into an X' if necessary, and prove that our aim is surely attained in repeating this a number of times.

Let $u_1, \cdots, u_m$ be an ε-net for ρ. Denote by X' the set of m^2n elements $u_jx_iu_k$ $(i=1, \cdots, n;\, j, k=1, \cdots, m)$. We will show that

$$(2)\qquad \text{Osc } X' \leqq \frac{2\varepsilon}{m^2}+\frac{m^2-1}{m^2}\text{ Osc } X.$$

To the purpose, note that 1°) $\mu(X')=\frac{1}{m^2n}\sum_{i,j,k} f(u_jx_iu_k)=\frac{1}{m^2}\sum_{j,k}\mu(u_jXu_k)$ and that 2°) $\rho(a, a')<\varepsilon$, $\rho(b, b')<\varepsilon$ implies $|\mu(aXb)-\mu(a'Xb')|<2\varepsilon$ for any X. Indeed, $|\mu(aXb)-\mu(a'Xb')|=\left|\frac{1}{n}\sum\big(f(ax_ib)-f(a'x_ib')\big)\right|\leqq\left|\frac{1}{n}\sum\big(f(ax_ib)-f(a'x_ib)\big)\right|+\left|\frac{1}{n}\sum\big(f(a'x_ib)-f(a'x_ib')\big)\right|$, and $|f(ax_ib)-f(a'x_ib)|\leqq\rho(a, a')$, $|f(a'x_ib)-f(a'x_ib')|\leqq\rho(b, b')$.

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1) More adequately, it would be called an "*invariant* mean value."

Now $\mu(aX'b)-\mu(a'X'b')=\frac{1}{m^2}\left(\sum_{j,k}\mu(au_jXu_kb)-\sum_{j,k}\mu(a'u_jXu_kb')\right)$ by 1°), and as $u_1, \dots, u_m$ is an ε-net, there exist certain j_0, k_0, so that $\rho(au_1, a'u_{j_0})<\varepsilon$, $\rho(u_1b, u_{k_0}b')<\varepsilon$. The right-hand side of the last equation $=\frac{1}{m^2}\left(\mu(au_1Xu_1b)-\mu(a'u_{j_0}Xu_{k_0}b')\right)+\frac{1}{m^2}\left(\sum_{(j,k)\neq(1,1)}-\sum_{(j,k)\neq(j_0,k_0)}\right)$. In evaluating this in taking account of 2°), we obtain (2).

Let $X^{(\nu)}$ be the set obtained from X after operating ν times the process $X\rightarrow X'$. We have then by (2)

$$\text{Osc } X^{(\nu)} \leqq 2\varepsilon+\left(\frac{m^2-1}{m^2}\right)^{\nu}(\text{Osc } X-2\varepsilon).$$

Either Osc X is already $\leqq 2\varepsilon$, or else Osc $X^{(\nu)}$ becomes $<3\varepsilon$, say, for a sufficiently large ν. In any way, we get Osc $X_1<3\varepsilon$ for a certain X_1; $\mu(X_1)$ is then clearly a 3ε-appr. m. v. of f. In virtue of what we have seen in the last paragraph, we have established herewith the

Theorem III. Every a.p.f. has a unique m. v.

The m. v. of f is denoted by Mf. The known properties of this functional are to show in the usual way; in particular, the linearity: $M(\alpha f+\beta g)=\alpha Mf+\beta Mg$ is an immediate consequence of the lemma 5.[1]

1) See J. v. Neumann, l. c.