# 38. *On Affine Geometry of Abelian Groups.*

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1. Let a set $G$ of elements $a, b, c, \dots,$ satisfy the following axioms:

(1) For any pair of elements $a, b$, the product $a\cdot b$ of $G$ determines uniquely one element $c$ in $G$, viz. $a\cdot b=c$.

(2) For two given elements $a$ (or $b$) and $c$, the equation $a\cdot b=c$ can be uniquely solved by $b$ (or $a$) in $G$.

(3) For any four elements $a, b, c$ and $d$, we have

$$(a\cdot b)(c\cdot d)=(a\cdot c)\cdot(b\cdot d).$$

(4) Each element $a$ in $G$ is idempotent, viz. $a\cdot a=a$.

In my previous paper[1], we know that the operation $a\cdot b$ represents an abstract generalization of the mean operation which divides the straight line joining two points $a, b$ in a given ratio $m:n$.

Now, we apply

*Definition* ($A$). *Let us suppose that a set $L(a, b)$ consists of elements which are produced from two elements $a, b$ by the operation (1) and its inverse operation (2) with possible repetitions. Then, we say that the set $L(a, b)$ is the straight line*[2] *joining two points $a, b$ in the space $G$.*

Under the above definition ($A$), can we constitute a space of affine geometry from $G$?[3] But the answer for this problem is not always affirmative, because the straight line $L(a, b)$ of $G$ does not necessarily admit the familiar proposition:

($L$) *Any two straight lines, not parallel to each other, meet in one and only one point.*

In the following lines, we shall proceed to find a necessary and sufficient condition of the problem.

2. In place of the product $a\cdot b$, let us introduce the new product $a+b$ into $G$ as follows:

(5) Let $a$ and $b$ be any two given elements in $G$. If $a=x\cdot s$ and $b=s\cdot y$, for a fixed element $s$, then we put $a+b=x\cdot y$.

Here, we know[4] that the set $G$ forms an abelian group with respect to the new product $a+b$ and moreover the old product $a\cdot b$

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1) K. Toyoda, On Axioms of Mean Transformation and Automorphic Transformations of Abelian Groups, Tôhoku Math. Journal **47** (1940).

2) This definition is due to the remark of M. Takasaki.

3) G. Hessenberg, Acta Math. 29;
H. Wiener, Jahrsber. d. D. M. V. (1891);
G. Hassenberg, Grundlagen der Geometrie;
M. Pasch und M. Dehn, Vorlesungen über neue Geometrie;
K. Reidmeister, Vorlesungen über Grundlagen der Geometrie;
O. Veblen and J. W. Young, Projective Geometry, I, II;
H. Weyl, Raum, Zeit, Matrie.

4) K. Toyoda, loc. cit.

can be expressed as a linear form of $a$ and $b$, say, $a\cdot b=Aa+(E-A)b$, provided that $A$ and $E-A$[1] denote two automorphic transformations of the abelian group $G$.

Now, putting

(6) $a\cdot b=Aa+(E-A)\,b=A\,(a,b)$,

we have the relations:

(7) $A\big(B(a,b),b\big)=AB(a,b)$,

(8) $C\big(A(a,b),B(a,b)\big)=\big(CA+(E-C)B\big)\ (a,b)$,

(9) $A(a,b)=(E-A)\ (b,a)$,

(10) If $A(a,b)=c$, then $a=A^{-1}\ (c,b)$ and $b=(E-A)^{-1}\ (c,a)$.

First we have

*Theorem 1.* If we put

(11) $$L(a,b)=b+R(A)\ (a-b),$$

$R(A)$ forms a ring of automorphic transformations and contains the identical transformation $E$.

*Proof.* By making use of the above relations (6), (7), (8), (9) and (10), it is evident that we may put

$$L(a,b)=b+R(A)\ (a-b).$$

Next, we shall prove that $R(A)$ forms a ring containing $E$ as follows:

(i) From (7), we get

$$BC\in R(A),\quad \text{for any elements } B, C \text{ in } R(A).$$

(ii) If $B$, $C$ are contained in $R(A)$, then we obtain, by means of (7) and (10),

$$A^{-1}B\in R(A) \quad \text{and} \quad (E-A)^{-1}C\in R(A),$$

so that, by using (8),

$$A(A^{-1}B)+(E-A)\ \big((E-A)^{-1}C\big)=B+C\in R(A).$$

(iii) From (7), we have

$$A^2\in R(A),$$

whence, with the help of (9),

$$(E-A^2)=(E-A)\ (E+A)\in R(A).$$

Consequently, we get from (7) and (10),

$$(E-A)^{-1}\ \big((E-A)\ (E+A)\big)=E+A\in R(A),$$

from which, by means of (9),

$$E-(E+A)=-A\in R(A).$$

1) $E$ denotes the identical transformation.

(iv) Also, it is evident by means of (6) and (10),

$$A^{-1}A=E\in R(A).$$

3. Furthermore, we shall prove the following

*Lemma.* In order that the straight line $L(a, b)$ of $G$ admits the proposition $(L)$, it is necessary and sufficient that $R(A)$ forms a corpus of automorphic transformations.

*Proof.* By means of (11), it is self-evident.

*Theorem 2.* In order that $G$ constitutes a space of affine geometry under the definition $(A)$, it is necessary and sufficient that $G$ admits the postulate:

(12) If $c$ and $d$ be any two distinct elements in

$$L(a, b)\,, \quad \text{then} \quad L(c, d)=L(a, b)\,.$$

*Proof.* Since the postulate (12) is equivalent to the proposition $(L)$, it is self-evident.

Accordingly, we have

*Theorem 3.* Let $G$ be a finite set[1]. Then, in order that $G$ constitutes a space of affine geometry under the definition $(A)$ for a proper automorphic transformation $A$, it is necessary and sufficient that $G$ becomes a direct product of cyclic sub-groups of order $p$[2] with respect to the new product $a+b$, that is to say, the invariant of $G$ is $(p, p, \cdots, p)$.

*Proof.* By means of Lemma, $R(A)$ contains $E$. Therefore, in order that $R(A)$ forms a corpus, $R(A)$ must contain a sub-corpus $\{\lambda E\}$ of characteristic $p$. Thus we can prove the theorem by the existence[3] of the sub-corpus $\{\lambda E\}$.

4. Finally, we shall give some remarks.

*Theorem 4.* In the definition $(A)$, let us replace the fixed element $s$ by another fixed element $t$, viz. if $a=z\cdot t$ and $b=t\cdot w$, then we put $a*b=z\cdot w$. Then, between the new products $a+b$ and $a*b$ there exists the relation $a*b=a+b-t$. Moreover[4], the fixed elements $s$ and $t$ become the unit elements of $G$ with regard to the new product $a+b$ and $a*b$ respectively.

*Proof.* By means of (6) and (10), it follows that

$$z=A^{-1}(a, t) \quad \text{and} \quad w=(E-A)^{-1}(b, t)\,,$$

so that

$$\begin{aligned} a*b&=z\cdot w \\ &=A\big(A^{-1}(a, t)\big)+(E-A)\left((E-A)^{-1}(b, t)\right) \\ &=a+A(E-A^{-1})\,t+b+(E-A)\left(E-(E-A)^{-1}\right)t \\ &=a+b-t\,. \end{aligned}$$

1) O. Veblen and W. H. Bussey, Finite Projective Geometries,
I, Trans. Am. Math. Soc., **7** (1906), pp. 241–259;
II, Trans. Am. Math. Soc., **8** (1907), pp. 266–268.

2) $p$ denotes a prime number.

3), 4) K. Toyoda, loc. cit.

*Remark 1.* In the case $G$ constitutes an $n$-dimensional euclidean space and $A=\lambda E$, for a real number $\lambda$, $L(a, b)$ is a straight line of one-dimension. In general, in the case $A \neq \lambda E$, $L(a, b)$ is not always a straight line of one-dimension, but $m$-dimensional hyper-plane.

*Remark 2.* In the case where we consider a set of systems of axioms (1), (2), (3), and (4) in place of one system of axioms, we can proceed in the same way as before.