PAPERS COMMUNICATED

1. An Abstract Integral, IV.

By Shin-ichi IZUMI.

Mathematical Institute, Tohoku Imperial University, Sendai.

(Comm. by M. FUJIWARA, M.I.A., Jan. 13, 1941.)

In the integral defined in this paper, the concept of "function" is dropped and that of "partition" is abstracted. The "abstract partition" contains as particular cases closure operations and transformations appearing in the ergodic theory. And then thus developed integration theory contain closure topology and ergodic theory.

1. Let $\mathbb{L}$ be a vector lattice,[1] that is, $\mathbb{L}$ is a linear space and the relation "$f \geqq 0$" is defined for some $f \in \mathbb{L}$ such as ($f, g, \dots$ are elements of $\mathbb{L}$ and $\lambda, \mu, \dots$ are real numbers)

($\mathbb{L}$, 1) if $f \geqq 0$ and $\lambda \geqq 0$, then $\lambda f \geqq 0$,

($\mathbb{L}$, 2) if $f \geqq 0$ and $g \geqq 0$, then $f+g \geqq 0$,

and further, if $f \geqq g$ is defined to mean $f-g \geqq 0$, then

($\mathbb{L}$, 3) the relation "$f \geqq g$" defines a lattice.

Let $\mathbb{T}$ be a certain space and $\{T^a\}$ be a set of transformations from $\mathbb{L}$ to $\mathbb{T}$, such that

(T, 1) $\{T^a\}$ is the Moore-Smith set or index set $\{a\}$ has the Moore-Smith property, that is for any two α and β in $\{a\}$, there exists a γ such that $\gamma \geqq \alpha$ and $\gamma \geqq \beta$ (or $\gamma \leqq \alpha$ and $\gamma \leqq \beta$), where "$\leqq$" is a transitive asymmetric relation. Further axioms for $\mathbb{T}$ and $\{T^a\}$ will be added later.

Example 1. (L) (the space of integrable functions in $(0, 1)$) satisfies the axioms of $\mathbb{L}$. If we take $\mathbb{T} = R_1$ (set of all real numbers) and T^a as the operation taking Lebesgue sum, then xT^a becomes the Lebesgue sum of function x in (L). If the limit $\lim_a xT^a$ is defined in ordinary manner, then it becomes the Lebesgue integral of x.

Starting from (M) (set of all real measurable functions) which satisfies also the axioms of $\mathbb{L}$, we get (L) as the set of all elements x such that $\lim_a xT^a$ exists.

Example 2. Let $\mathbb{L} = \mathbb{T}$ and τ be a certain operator from $\mathbb{L}$ onto itself. Let $\tau^1 = \tau$, $\tau^n = \tau(\tau^{n-1})$ $(n > 1)$ and put $fT^n = \{f\tau + \cdots + f\tau^n\}/n$. We define $\lim_n xT^n$. If the limit exists for an f and is equal to the "integral" of x, then such f is called ergodic or ergodic element.

Example 3. Let E be the set of all characteristic functions and their linear combinations on a certain topological space. E satisfies the axioms of $\mathbb{L}$. We take $\mathbb{T}$ as E itself and T^a as a closure operation. That is, for a characteristic function e in E, eT means closure in the ordinary sense and for non-characteristic function $f = \sum_{i=1}^{n} \alpha_i e_i$ (e_i being

---

1) We use the terminologies in Birkhoff, Lattice Theory.

characteristic functions) we define $fT^a=\sum_{i=1}^{n} a_i e_i T^a$. Thus all axioms are satisfied.

2. First suppose that

$(\mathbb{T}, 1)$ $\mathbb{T}$ is a (conditionary) σ-complete vector lattice.

In $\mathbb{T}$, we introduce a limit called relative uniform convergence. That is, Moore-Smith set $\{x^a\}$ is relative uniformly convergent to x if and only if there exist a sequence $\{\lambda_a\}$ of real numbers and an element $u \in \mathbb{T}$ such that $|x_a - x| \leqq \lambda_a u$, and $\lambda_a \geqq \lambda_\beta$ if $\beta > \alpha$ and for any $\varepsilon > 0$ and λ_a, there exists λ_β such as $\beta > \alpha$ and $|\lambda\gamma| < \varepsilon$ for all $\gamma > \beta$.

If $\{x_a\}$ converges to x relative uniformly, then the limit is unique. We write $\mathrm{Lim}_a\, x_a = x$ or $x_a \to x$. For an f in $\mathbb{L}$, if the limit $\mathrm{Lim}_a\, fT^a$ exists, then we write $\mathrm{Lim}_a\, fT^a = fT$ or $\int f$. By L we denote the entity of such f in $\mathbb{L}$.

Let us suppose that

$(T, 2)$ each T^a is linear, that is, $(\lambda f + \mu g)T^a = \lambda(fT^a) + \mu(gT^a)$ for any real λ, μ and any f, g in $\mathbb{L}$.

$(T, 3)$ each T^a is positive, that is, $f \geqq 0$ implies $fT^a \geqq 0$.

$|x_a - x| \leqq \lambda_a u$ and $|y_a - y| \leqq \mu_a v$ imply $|(x_a + y_a) - (x + y)| \leqq (\lambda_a + \mu_a)(u \cup v)$. That is, $x_a \to x$ and $y_a \to y$ imply $x_a + y_a \to x + y$. Therefore $(T, 2)$ and $(T, 3)$ give us

$(L, 1)$ $f \in L$ and $g \in L$ imply $\lambda f + \mu g \in L$,

$(L, 2)$ $f \in L$ and $f \geqq 0$ imply $fT \geqq 0$ and $(\lambda f + \mu g)T = \lambda(fT) + \mu(gT)$.

We have also

$(L, 3)$ $f \in L$ implies $|f| \in L$.

For, since $\mathbb{L}$ is a vector lattice, $f = f^+ + f^-$, and then $fT^a = f^+T^a + f^-T^a$. By $(\mathbb{T}, 1)$ $(fT^a)^+ = f^+T^a$. $|f^+T^a - (fT)^+| \leqq |fT^a - fT|$ and $\mathrm{Lim}_a\, fT^a = fT$ imply that $\mathrm{Lim}_a\, f^+T^a$ exists and is equal to $(fT)^+$, that is $f^+T = (fT)^+$. Similarly f^-T and then $|f|\,T$ exist. Thus we get the theorem.

3. If $\{x_n\}$ is a sequence in $\mathbb{L}$ or $\mathbb{T}$ and $\{x_n\}$ converges to x by the order topology, then we say that $\{x_n\}$ is (o)-convergent to x and write $\lim_n x_n = x$ or $x_n \to x(o)$. lim sup and lim inf will be defined by the order topology.

If $\mathrm{Lim}_a\, fT^a$ exists, then we can find confinal (enumerable) sequence $\{T^{a_k}\}$ such that $a_k < a_{k+1}$ $(k=1, 2, 3, \ldots)$ $\mathrm{Lim}_k\, fT^{a_k} = \mathrm{Lim}_k\, fT^a$.

Let $\{f_n\}$ be a sequence in $\mathbb{L}$. If $\mathrm{Lim}_a\, f_nT^a$ $(n=1, 2, 3, \ldots)$ exist, then by the diagonal method we can select a confinal sequence $\{T^{a_k}\}$ $(k=1, 2, 3, \ldots)$ such that $a_k < a_{k+1}$ $(k=1, 2, 3, \ldots)$ and $\mathrm{Lim}_k\, f_nT^{a_k} = \mathrm{Lim}\, f_nT^a$ for each n. Such $\{T^{a_k}\}$ is called confinal sequence with respect to $\{f_n\}$.

We will suppose that

$(T, 4)$ $\{T^a\}$ is "sequencially closed," that is, for any sequence $\{f_n\} \subset L$ such as $\lim_n f_n = f$ if the limit $\lim_k f_kT^{a_k} = f^*$ exists for any confinal sequence $\{T^{a_k}\}$ with respect to $\{f_n\}$, then $fT = f^*$.

Then we have

$(L, 4_1)$ If $\{f_n\} \subset L$, $f_n \geqq 0$, $f=\sum_{n=1}^{\infty} f_n$ (in order topology) and $\sum_{n=1}^{\infty} f_n T$ converges in order topology, then $f \in L$ and $fT=\sum_{n=1}^{\infty} f_n T$.

For, if $\{T^{a_k}\}$ is a confinal sequence with respect to $\{f_n\}$, then

$$\sum_{i=1}^{m} f_i T^{a_{k,i}} \leqq \sum_{i=1}^{k} f_i T^{a_{k,i}} \leqq \sum_{i=1}^{\infty} f_i T^{a_{k,i}}$$

where $k \geqq m$ and $\{a_{k,i}\}$ is a subsequence of $\{a_k\}$ such as $a_k \leqq a_{k,i}$. Let $k \to \infty$, then

$$(1) \qquad \sum_{i=1}^{m} f_i T \leqq \liminf_k \sum_{i=1}^{k} f_i T^{a_{k,i}} \leqq \limsup_k \sum_{i=1}^{k} f_i T^{a_{k,i}}$$

$$\leqq \sum_{i=1}^{\infty} f_i T + \limsup_k \sum_{i=1}^{\infty} |f_i T^{a_{k,i}} - f_i T| .$$

Now we can find $a_{k,i}$ such that $\sum_{i=1}^{\infty} |f_i T^{a_{k,i}} - f_i T| \leqq (1/m)u$ provided that $\mathbb{T}$ satisfies the following axiom:

$(\mathbb{T}, 2)$ for any sequence u^n in $\mathbb{T}$, there exists a sequence of real numbers such as $\sum_{n=1}^{\infty} \lambda_n u^n$ converges in order topology.

Letting $m \to \infty$ in (1), we see that the limit $\lim_k \sum_{i=1}^{k} f_i T^{a_{k,i}}$ and then $\lim (\sum_{i=1}^{k} f_i) T^{\beta_k}$ (for suitable β_k) exist and

$$\lim_k (\sum_{i=1}^{k} f_i) T^{\beta_k} = \sum_{i=1}^{\infty} f_i T .$$

Therefore $(T, 4)$ gives the required result.

$(L, 4)$ If $0 \leqq f_n \leqq f_{n+1}$ and $\lim_n f_n = f$ where $f_n \in L$ and $\lim_n f_n T$ exists, then $f \in L$ and $\lim_n f_n T = fT$.

For, if we put $g_n = f_{n+1} - f_n$, then $f = f_1 + \sum_{n=1}^{\infty} g_n$ where $f_1 \geqq 0$, $g_n \geqq 0$ $(n=1, 2, 3, \ldots)$ and $g_n \in L$. By $(L, 4_1)$ we get

$$fT = f_1 T + \sum_{n=1}^{\infty} g_n T = \lim_k (f_1 T + \sum_{n=1}^{k} g_n T) = \lim_k f_k T . \qquad \text{q. e. d.}$$

$(L, 5_1)$ If $\{f_n\} \subset L$ and $f_n \geqq 0$, then

$$(\liminf_n f_n) T \leqq \liminf_n (f_n T) .$$

For, if we put $g_i = \bigwedge_{n=1}^{\infty} f_n$ $(i=1, 2, 3, \ldots)$ then g_i decreases and converges to $g = \liminf_n f_n$. By $(L, 4)$ $gT = \lim_n (g_n T) \leqq \liminf_n (f_n T)$, where the last two terms evidently exist.

$(L, 5)$ If $f_n \in L$, $\lim_n f_n = f$ and there is a $g \in L$ such that $|f_n| \leqq g$ $(n=1, 2, 3, \ldots)$, then $f \in L$ and $\lim_n f_n T = fT$.

For, since $g - f_n \geqq 0$, $g - f_n \in L$ and $\lim_n (g - f_n) = g - f$, we have by

$(L, 5_1)$ $(g-f)T \leqq \liminf_n (g-f_n)T$. Considering $g+f_n$ instead of $g-f_n$, we get similarly $(g+f)T \leqq \liminf_n (g+f_n)T$. These relations give us

$$\liminf_n (f_nT) \leqq fT \leqq \liminf (f_nT). \qquad \text{q. e. d.}$$

4. Thus we have reached the following theorem:

Theorem 1. If $\mathbb{L}$ and $\mathbb{T}$ are vector lattices satisfying the axiom $(\mathbb{T}, 2)$ the latter being conditionally σ-complete and $\{T^a\}$ is the Moore-Smith set of positive, additive and sequencially closed transformations from $\mathbb{L}$ onto $\mathbb{T}$, then $\{T^a\}$ defines a Lebesgue integral T and the class L of integrable functions, that is, for each $f \in L$ there corresponds $fT \in \mathbb{T}$ which has the properties $(L, 1)$, $(L, 2)$, $(L, 3)$, $(L, 4)$ and $(L, 5)$.

5. Next we suppose that $\mathbb{T}$ is a Banach lattice and replace relative uniform convergence by the norm convergence which is defined as follows: Moore-Smith set $\{x_a\}$ is convergent to x in the norm topology if and only if, for any $\varepsilon > 0$ and x_a, there exists an x_β such that $\|x-x_\gamma\| < \varepsilon$ for all $\gamma > \beta$. Then we get the analogous theorem to Theorem 1, which will be called *Theorem 2.*

On the other hand, supposing $\mathbb{T}$ as (conditionally) σ-complete vector lattice if we use order convergence instead of relative uniform convergence, then we get also the theorem analogous to Theorem 1, which will be called *Theorem 3.*

6. In the ergodic theory relative uniform convergence corresponds to dominated convergence, norm convergence to mean convergence and order convergence to almost everywhere convergence. Thus our theorems indicate that time average converges to space average and Theorem 1, 2 and 3 correspond to the Wiener-Fukamiya theorem, von Neumann-Yosida theorem and the G. D. Birkhoff theorem respectively.