43. On a Metric Connection Along a Curve in a Special Kawaguchi Space of Order Two.

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The foundation of the geometry in a special Kawaguchi space with the arc length $s=\int\left\{A_i(x, x')x''^i+B(x, x')\right\}^{\frac{1}{p}}dt$ has been studied by Prof. A. Kawaguchi[1) 2)].

In the present paper, it is proposed to introduce a metric connection along a curve in the same space according to Kawaguchi's theory. The same notations are adopted here as those by Prof. A. Kawaguchi.

1. In one of his previous papers Prof. A. Kawaguchi[2)] has introduced the covariant derivative by the parameter t, under the assumption that $p \neq 1$:

$$(1) \qquad \delta^* v^k = \frac{dv^k}{dt} + M^{k}_{\cdot i} v^i + L^{k}_{\cdot li} x''^l v^i ,$$

where $M^{k}_{\cdot i}$ is an object and $L^{k}_{\cdot li}$ is a tensor which are homogeneous of degree 1 and -1 with regard to the x'^j respectively, and the latter is symmetric with respect to both the undersubscripts.

Let x'^k be put into (1) instead of v^k and the left-hand side be denoted by $\check{x}^{[2]k}$, then it is found

$$(2) \qquad \check{x}^{[2]k} = q^k_l x''^l + M^{k}_{\cdot i} x'^i$$

putting

$$q^k_l = \delta^k_l + L^{k}_{\cdot li} x'^i ,$$

which is a tensor homogeneous of degree zero with regard to the x'^j.

Making use of the tensor Q^l_k such that $Q^l_k q^k_m = \delta^l_m$ under the assumption $p \neq \frac{5}{2}$, $|q^k_l| \neq 0$ (2) offers

$$(3) \qquad x''^h = Q^h_k \check{x}^{[2]k} - Q^h_k M^{k}_{\cdot i} x'^i .$$

Putting (3) into (1), one obtains

$$(4) \qquad \delta^* v^k = \frac{dv^k}{dt} + (M^{k}_{\cdot j} - L^{k}_{\cdot lj} Q^l_m M^{m}_{\cdot\cdot i} x'^i) v^j + L^{k}_{\cdot lj} Q^l_i \check{x}^{[2]i} v^j ,$$

which defines a covariant derivative along a curve in the space.

1) A. Kawaguchi, Geometry in an n-dimensional space with the arc length $s=\int\left\{A_i(x, x')x''^i+B(x, x')\right\}^{\frac{1}{p}}dt$, Transactions of the A. M. S., **44**, No. 2 (1938).

2) A. Kawaguchi, Die Geometrie des Integrals $\int(A_i x''^i+B)^{\frac{1}{p}}dt$, Proc. **12** (1936), 205–208.

But this is not invariant by a transformation of the parameter t. In the following there will be determined an intrinsic (i.e. independent of any choice of the parameter t) connection along the curve.

It is well known[1] that the tensor

$$I^{k}_{\cdot lj}=\frac{1}{3}H^{ki}\big(A_{i(l)(j)}+A_{l(i)(j)}+(p-3)A_{j(i)(l)}\big)$$

is homogeneous of degree -1 with regard to the x'^{j}, holding the following relation:

$$I^{h}_{\cdot lj}x'^{j}=0\,. \tag{5}$$

Let us consider

$$\delta v^{k}=\frac{dv^{k}}{dt}+(M^{k}_{\cdot j}-L^{k}_{\cdot lj}Q^{l}_{m}M^{m}_{\cdot i}x'^{i})\,v^{j}+I^{k}_{\cdot jl}Q^{l}_{i}\overset{\vee}{x}{}^{[2]i}v^{j} \tag{6}$$

which is obtained from (4) taking $I^{k}_{\cdot lj}$ for $L^{k}_{\cdot lj}$, then the right-hand side of the last expression is also a vector.

The vector $Q^{l}_{i}\overset{\vee}{x}{}^{[2]i}$ is varied by a transformation of parameter $\bar{t}=\bar{t}(t)$ in the way

$$(Q^{l}_{i}\overset{\vee}{x}{}^{[2]i})_{\bar{t}}=\alpha^{2}Q^{l}_{k}\overset{\vee}{x}{}^{[2]k}+x'^{l}\alpha' \tag{7}$$

where

$$\alpha=\frac{dt}{d\bar{t}}\,,\qquad \alpha'=\frac{d\alpha}{d\bar{t}}\,.$$

It will be easily seen from (5) and (7) that (6) is intrinsic, accordingly it defines the desired connection.

It is possible to give (6) in another form, that is

$$\delta v^{k}=\frac{dv^{k}}{dt}+J^{k}_{\cdot ij}x'^{i}v^{j}+I^{k}_{\cdot ij}x''^{i}v^{j} \tag{8}$$

where

$$J^{k}_{\cdot ij}=M^{k}_{\cdot i(j)}+(I^{k}_{\cdot li}-L^{k}_{\cdot li})\,Q^{l}_{m}M^{m}_{\cdot j}\,.$$

2. In order to determine an intrinsic metric tensor use will be made of the covariant derivative in $K^{(1)}_{n}$ defined by Prof. A. Kawaguchi under the assumption $p\neq\frac{3}{2}$:

$$Dv^{i}=\frac{dv^{i}}{dt}+\Gamma^{i}_{(j)}v^{j}\,. \tag{9}$$

It is known[2] that by a change of parameter $\bar{t}=\bar{t}(t)$ the scalar $x^{[2]i}DA_{i}$ is transformed into

$$(x^{[2]i}DA_{i})_{\bar{t}}=\alpha^{p+1i}x^{[2]i}DA+(p-3)\,F\alpha^{p-1}\alpha', \tag{10}$$

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1) A. Kawaguchi, Geometry in an n-dimensional space etc., loc. cit., p. 165 (2.12).

2) A. Kawaguchi, loc. cit., p. 160.

and the Synge vector into

$$(11)\qquad \left(\frac{1}{F^{\frac{p-1}{p}}}T_i\right)_{\bar t}=\frac{1}{F^{\frac{p-1}{p}}}T_i-(2p-3)\frac{1}{F^{\frac{p-1}{p}}}A_i\frac{\alpha'}{\alpha^2}\,.$$

It can be seen from (10) that the vector $\frac{2p-3}{p-3}\frac{1}{F^{\frac{p-1}{p}}}A_ix^{[2]j}DA_j$ behaves under a change of parameter $\bar t=\bar t(t)$ as follows

$$(12)\qquad \left(\frac{2p-3}{p-3}\frac{1}{F^{\frac{p-1}{p}}}A_ix^{[2]j}DA_j\right)_{\bar t}=\frac{2p-3}{p-3}\frac{1}{F^{\frac{2p-1}{p}}}A_ix^{[2]j}DA_j$$

$$+(2p-3)\frac{1}{F^{\frac{p-1}{p}}}A_i\frac{\alpha'}{\alpha^2}\,,$$

when $p\neq 3$.

From (11) and (12) is derived the intrinsic covariant vector

$$(13)\qquad \mathfrak{S}_i=\frac{1}{F^{\frac{p-1}{p}}}\left(T_i+\frac{2p-3}{p-3}x^{[2]j}\frac{A_iDA_j}{F}\right),$$

which satisfies

$$\mathfrak{S}_ix'^i=pF^{\frac{1}{p}}\,.$$

Let the intrinsic symmetric tensor g_{ij} be defined by

$$(14)\qquad g_{ij}=\frac{1}{F^{\frac{p-3}{p}}}(A_{i(j)}+A_{j(i)})+\frac{1}{p^2}\mathfrak{S}_i\mathfrak{S}_j$$

for which the next relation holds good:

$$(15)\qquad g_{ij}x'^ix'^j=F^{\frac{2}{p}}\,.$$

Now the arc length of the curve in the space is given by $s=\int\sqrt{g_{ij}x'^ix'^j}\,dt$ as well as in a Riemannian space.

By this reason this tensor can be taken as the metric tensor in the space and its contravariant components are defined by $g_{ij}g^{ik}=\delta^k_j$, when $|g_{ij}|\neq 0$.

In conclusion an intrinsic metric connection can be established along a curve in our space, applying Kawaguchi's method[1] to (8)

$$(16)\qquad \theta v^k=\delta v^k+\frac{1}{2}g^{kh}\delta g_{hi}v^i\,,$$

or

$$(17)\qquad \theta v^k=\frac{dv^k}{dt}+\overset{0}{\Lambda}{}^k_{ji}x'^jv^i+\overset{1}{\Lambda}{}^k_{ji}x''^jv^i+\overset{2}{\Lambda}{}^k_{ji}x'''^jv^i\,,$$

1) A. Kawaguchi, Beziehung zwischen einer metrischen linearen Übertragung und einer nicht-metrischen in einem allgemeinen metrischen Raume, Proc. Amst., **40** (1937), 596-601.

where

$$\overset{0}{\Lambda}{}^{i}_{jk}=\{{}^{\,i}_{jk}\}_0+\frac{1}{2}g^{ih}\left\{J^{l}_{\cdot jk}g_{lh}-J^{l}_{\cdot hk}g_{lj}-\frac{\partial g_{hk}}{\partial x^{j}}+\frac{\partial g_{jk}}{\partial x^{h}}\right\},$$

$$\overset{1}{\Lambda}{}^{i}_{jk}=\{{}^{\,i}_{jk}\}_1+\frac{1}{2}g^{ih}\left\{I^{l}_{\cdot jk}g_{lh}-I^{l}_{\cdot hk}g_{lj}-\frac{\partial g_{hk}}{\partial x'^{j}}+\frac{\partial g_{jk}}{\partial x'^{h}}\right\},$$

$$\overset{2}{\Lambda}{}^{i}_{jk}=\frac{1}{2}g^{ih}\frac{\partial g_{hj}}{\partial x''^{k}},$$

$$\{{}^{\,i}_{jk}\}_a=\frac{1}{2}g^{ih}\left(\frac{\partial g_{hj}}{\partial x^{(a)k}}+\frac{\partial g_{hk}}{\partial x^{(a)j}}-\frac{\partial g_{jk}}{\partial x^{(a)h}}\right).$$

By virtue of (16) or (17) one can develop the theory of the curve in the space similarly as in Riemannian space.

It is to be noticed that the above theory is applicable for every value of p but for 1, $\frac{3}{2}$, $\frac{5}{2}$, 3.