# 42. A Symmetric Connection in an n-dimensional Kawaguchi Space.

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*Introduction.* Geometry in the manifold, in which the arc length $s$ of a curve $x^i=x^i(t)$ is given by $s=\int\left\{A_i(x,x')x''^i+B(x,x')\right\}^{\frac{1}{p}}dt$, was first established by Prof. A. Kawaguchi[1]. In his work two kinds of connections $C$ and $C'$ are introduced. The present author proposes to introduce another connection $\mathfrak{C}$ in this manifold $K_n^{(1)}$.

§ 1 is devoted to the exposition of the various quantities in the manifold $K_n^{(1)}$ and § 2 to the establishment of the symmetric connection $\mathfrak{C}$ in our manifold. In § 3 the curvature and torsion tensors are calculated. The symbolism employed in this paper is similar to that of Prof. A. Kawaguchi[1].

1. *Exposition of the various quantities.* One starts with an $n$-dimensional manifold with coordinates $x^i$ in which the arc length of a curve $x^i=x^i(t)$ is given by the integral

$$s=\int\left\{A_i(x,x')x''^i+B(x,x')\right\}^{\frac{1}{p}}dt. \tag{1.1}$$

It is supposed that $A_i$ and $B$ are analytic in a certain region of their arguments and the arc length remains unaltered by any transformation of the parameter $t$. The latter condition implies the following identities in $x^i$ and $x'^i$:

$$\begin{cases} A_ix'^i=0, \\ A_{k(i)}x'^i=(p-2)A_k, \qquad B_{(i)}x'^i=pB, \end{cases} \tag{1.2}$$

where partial differentiation by $x'^i$ and $x^i$ is denoted with $(i)$ and $(0)i$ respectively. Thus one concludes that the $A_i$ is homogeneous of degree $p-2$ with regard to the $x'^i$ and $B$ of degree $p$. From the first equation (1.2) it follows, on differentiating by $x'^i$, that

$$A_{i(k)}x'^i=-A_k. \tag{1.3}$$

Now one puts

$$F=A_ix''^i+B, \tag{1.4}$$

and introduces the Craig vector with respect to the function $F$:

$$-T_i=G_{ij}x''^j+2\Gamma_i, \tag{1.5}$$

where

$$G_{ij}=2A_{i(j)}-A_{j(i)}, \qquad 2\Gamma_i=2A_{i(0)l}x'^l-B_{(i)}. \tag{1.6}$$

1) A. Kawaguchi, Geometry in an $n$-dimensional space with the arc length $s=\int\left\{A_i(x,x')x''^i+B(x,x')\right\}^{\frac{1}{p}}dt$, Trans. Amer. Math. Soc., **44**, no. 2 (1938), 153-167.

Next it is assumed that $p \neq 3/2$ and that the determinant $|G_{ij}|$ does not vanish identically, then there are obtained the contravariant forms of (1.5):

$$x^{(2)i} = -T_l G^{li} = x''^i + 2\Gamma^i, \tag{1.7}$$

where

$$G_{il} G^{ij} = \delta_l^j, \qquad \Gamma^i = \Gamma_l G^{li}. \tag{1.8}$$

From (1.6) and (1.8) one gets the following relations:

$$A_i G^{il} = \frac{x'^l}{2p-3}, \qquad A_i G^{li} = -\frac{x'^l}{p}. \tag{1.9}$$

Finally one introduces a symmetric tensor of which the components are:

$$g_{ij} = A_{i(j)} + A_{j(i)}, \tag{1.10}$$

and assumes that the determinant $|g_{ij}|$ does not vanish identically. The case where $p \neq 3$ must be excluded, because the relation

$$g_{ij} x'^j = g_{ji} x'^j = (p-3) A_i \tag{1.11}$$

holds good.

2. *Determination of a symmetric connection.* From (1.7) it is seen that the expressions

$$Dx'^i = dx'^i + \Gamma^i_{(j)} dx^j \tag{1.12}$$

are the components of a vector. Hence it follows:

*Equations* (2.1) *define the base connections in the Kawaguchi space* $K_n^{(1)}$ *of order one and dimension n in which the arc length of a curve,* $x^i = x^i(t)$, *is given by the integral* (1.1)[1].

In order to introduce a symmetric connection in the manifold $K_n^{(1)}$, one sets the following three postulates:

(I) If $v^i(x, x')$ are the components of a contravariant vector and $v_i(x, x')$ those of a covariant vector, homogeneous of degree zero with regard to the $x'^i$. The covariant differentials corresponding to a displacement from a line element $(x, x')$ to a consecutive line element $(x+dx, x'+dx')$ are defined by the expressions:

$$\delta v^i = dv^i + \Gamma^i_{jk}(x, x') v^j dx^k \quad \text{and} \quad \delta v_i = dv_i - \Gamma^j_{ik}(x, x') v_j dx^k$$

respectively.

Making use of the base connection (2.1), one can write the first of the above equations in the form:

$$\delta v^i = \nabla_j v^i dx^j + \nabla'_j v^i Dx'^j,$$

where

$$\nabla_j v^i = v^i{}_{(0)j} - v^i{}_{(l)} \Gamma^l_{(j)} + \Gamma^i_{kj} v^k, \qquad \nabla'_j v^i = v^i{}_{(j)}, \tag{2.2}$$

1) See A. Kawaguchi, loc. cit.

which are defined as the covariant derivatives of the first and second kinds of a contravariant vector $v^i$. And the covariant derivatives of a covariant vector $v_i$ are defined analogously. Moreover, the covariant differential of a general tensor of which components are homogeneous of a certain degree $h$ with regard to the $x'^i$, has not been defined but its covariant derivatives of both kinds are also defined by analogy.

(II) The functions $\Gamma^i_{jk}(x, x')$ are symmetric with respect to two lower indices and analytic of their arguments.

(III) The covariant derivatives of the first kind of the tensor $g_{ij}$ vanish identically.

From the definition of the covariant derivative of the first kind (2.2) one has:

$$\nabla_k g_{ij} = g_{ij(0)k} - \Gamma_{ijk} - \Gamma_{jik} - g_{ij(l)}\Gamma^l_{(k)}, \tag{2.3}$$

where

$$g_{ij}g^{il} = \delta^l_j, \qquad \Gamma_{ijk} = g_{il}\Gamma^l_{jk}. \tag{2.4}$$

From the last postulate and (2.3) there is obtained the relation

$$\Gamma_{ikj} + \Gamma_{kij} = g_{ik(0)j} - g_{ik(l)}\Gamma^l_{(j)}. \tag{2.5}$$

If this equation is subtracted from the sum of the two equations obtained from it by interchanging $i$ and $j$ and $k$ and $j$ respectively and then divides both sides by two, there is obtained in consequence of the second postulate,

$$\Gamma_{ijk} = \gamma_{ijk} + \frac{1}{2}\{g_{ik(l)}\Gamma^l_{(j)} - g_{jk(l)}\Gamma^l_{(i)} - g_{ij(l)}\Gamma^l_{(k)}\}, \tag{2.6}$$

where the function $\gamma_{ijk}$ is Christoffel's symbol of the first kind with regard to $g_{ij}$. Finally if the index $j$ in (2.6) is raised by multiplication of $g^{jl}$, one gets

$$\Gamma^j_{ik} = \gamma^j_{ik} + \frac{1}{2}g^{ir}\{g_{ki(l)}\Gamma^l_{(r)} - g_{rk(l)}\Gamma^l_{(i)} - g_{ir(l)}\Gamma^l_{(k)}\}, \tag{2.7}$$

where $\gamma^i_{ik}$ are Christoffel's symbols of the second kind with regard to $g_{ij}$.

The coefficients of the connection $\Gamma^i_{jk}$ obtained above are symmetric with respect to the lower suffices $j$ and $k$ and contain only the first and second partial derivatives of the functions $A_i$ and $B$. The function $\Gamma^i_{jk}$ is transformed like an affine parameter of ordinary geometry under the transformation of coordinates but remains unaltered under the transformation of parameter $\bar{t} = \bar{t}(t)$.

3. *Curvature and torsion tensors.* Let $v^i(x, x')$ be any contravariant vector field in $K^{(1)}_n$ and its components homogeneous of degree zero with regards to the $x'^i$, then its covariant derivatives of the first and second kinds are given by the equations (2.2).

Computing the parenthesis of Poisson for the covariant derivatives, the curvature and torsion tensors are obtained as follows:

$$(3.1)\qquad (\nabla_j\nabla_k-\nabla_k\nabla_j)\,v^i=R_{jkl}^{\cdot\cdot\cdot i}v^l+K_{jk}^{\cdot\cdot l}\nabla'_l v^i\,,$$

$$(4.2)\qquad (\nabla_j\nabla'_k-\nabla'_k\nabla_j)\,v^i=-B_{jkl}^{\cdot\cdot\cdot i}v^l+C_{jk}^{\cdot\cdot l}\nabla'_l v^i\,,$$

where

$$(3.3)\qquad \begin{aligned} R_{jkl}^{\cdot\cdot\cdot i}=&\Gamma^i_{lk(0)j}-\Gamma^i_{lj(0)k}+\Gamma^m_{(k)}\Gamma^i_{lj(m)}-\Gamma^m_{(j)}\Gamma^i_{lk(m)}\\ &+\Gamma^i_{jm}\Gamma^m_{lk}-\Gamma^i_{km}\Gamma^m_{lj}\,,\end{aligned}$$

$$(3.4)\qquad K_{jk}^{\cdot\cdot l}=\Gamma^l_{(j)(0)k}-\Gamma^l_{(k)(0)j}+\Gamma^m_{(j)}\Gamma^l_{(k)(m)}-\Gamma^m_{(k)}\Gamma^l_{(j)(m)}\,,$$

$$(3.5)\qquad B_{jkl}^{\cdot\cdot\cdot i}=\Gamma_l{}^i{}_{j(k)}\,,$$

$$(3.6)\qquad C_{jk}^{\cdot\cdot l}=\Gamma^l_{(j)(k)}-\Gamma^l_{kj}\,.$$

One finds many relations satisfied by these tensors. They are

$$(3.7)\qquad \begin{cases} R_{jkl}^{\cdot\cdot\cdot i}+R_{kjl}^{\cdot\cdot\cdot i}=0\,, & R_{jkl}^{\cdot\cdot\cdot i}+R_{klj}^{\cdot\cdot\cdot i}+R_{ljk}^{\cdot\cdot\cdot i}=0\,,\\ B_{jkl}^{\cdot\cdot\cdot i}x'^k=0\,, & C_{jk}^{\cdot\cdot l}x'^j x'^k=0\,,\\ C_{jk}^{\cdot\cdot l}-C_{kj}^{\cdot\cdot l}=0\,, & K_{jk}^{\cdot\cdot l}+K_{kj}^{\cdot\cdot l}=0\,. \end{cases}$$

Further, it is not difficult to find the identities of Bianchi but they will not be calculated here.

Finally it is seen from the equations (3.6) that when the tensor $C_{jk}^{\cdot\cdot i}$ vanishes identically our connection $\mathfrak{C}$ is identical with that of Prof. A. Kawaguchi.