52. A Remark on the Theory of General Fuchsian Groups.

By Kiiti MORITA.

Tokyo Bunrika Daigaku, Tokyo.

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Prof. M. Sugawara has recently introduced a notion of general fuchsian groups and developed a theory of automorphic functions of higher dimensions[1]. In the present note we shall show that there is another class of groups which can be treated with his method. The classical case of hyperfuchsian groups is included here as a special one (the case $m=1$ below).

§ 1. *The space* $\mathfrak{A}_{(n,m)}$. *General thetafuchsian functions in* $\mathfrak{A}_{(n,m)}$. Let us consider the set $\mathfrak{R}_{(n,m)}$ of all matrices of the type (n, m). The subset of $\mathfrak{R}_{(n,m)}$, whose elements are matrices satisfying the condition $E^{(m)}-\bar{Z}'Z>0$[2], shall be denoted by $\mathfrak{A}_{(n,m)}$[3]. Now we put $S_{(n,m)}=\begin{pmatrix} E^{(n)} & 0 \\ 0 & -E^{(m)} \end{pmatrix}$. If a matrix U of order $(n+m)$ satisfies the condition

$$(1) \qquad \bar{U}'S_{(n,m)}U=S_{(n,m)},$$

then the substitution

$$(2) \qquad W=(U_1Z+U_2)(U_3Z+U_4)^{-1}$$

carries $\mathfrak{A}_{(n,m)}$ into itself, where $U=\begin{pmatrix} U_1 & U_2 \\ U_3 & U_4 \end{pmatrix}$, and the types of U_1, U_2, U_3, U_4 are respectively (n, n), (n, m), (m, n), (m, m). Hence the matrices satisfying the condition (1) induce the displacements in the space $\mathfrak{A}_{(n,m)}$ and form a group $\Gamma_{(n,m)}$. The matrices inducing the identical displacement in $\mathfrak{A}_{(n,m)}$ are of the form $\omega E^{(n+m)}$ $(|\omega|=1)$ and constitute a group $\Gamma^*_{(n,m)}$. The factor group $\Gamma_{(n,m)}/\Gamma^*_{(n,m)}$ is called the group $\mathfrak{B}_{(n,m)}$ of all displacements in $\mathfrak{A}_{(n,m)}$. $\mathfrak{B}_{(n,m)}$ is transitive in $\mathfrak{A}_{(n,m)}$: For a given point A we put

$$U_A=\begin{pmatrix} N^{-1} & -N^{-1}A \\ -M^{-1}\bar{A}' & M^{-1} \end{pmatrix}, \quad E^{(n)}-A\bar{A}'=N\bar{N}', \quad E^{(m)}-\bar{A}'A=M\bar{M}'.$$

Then U_A carries A into the zero point and $U_A \in \Gamma_{(n,m)}$.

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1) M. Sugawara, Über eine allgemeine Theorie der Fuchsschen Gruppen und Theta-Reihen, Ann. Math. **41**, 488–494; M. Sugawara, On the general Zetafuchsian functions, Proc. **16** (1940), 367–372; M. Sugawara, A generalization of Poincaré-space, Proc. **16** (1940), 373–377. In the sequel these papers will be cited as S. I, S. II, S. III respectively.

2) By $E^{(m)}$ we mean the unite matrix of order m. $H>0$ means that a hermitian matrix H is positive definite. The same notations as in S. I will be used in this note.

3) If we define the distance between two points Z_1 and Z_2 as $[Sp\overline{(Z_1-Z_2)}'(Z_1-Z_2)]^{\frac{1}{2}}$ then $\mathfrak{A}_{(n,m)}$ is an open, bounded, convex set in a complete metric space $\mathfrak{R}_{(n,m)}$.

The subgroups of $\mathfrak{B}_{(n,m)}$ without infinitesimal transformations are called general fuchsian groups. Since a matrix $U\in\Gamma_{(n,m)}$ fixing the zero point is of the form $\begin{pmatrix} U_1 & 0 \\ 0 & U_4 \end{pmatrix}$ with unitary matrices U_1 and U_4, the group of all displacements which leave the zero point unchanged is compact. Hence we have

Theorem: Every fuchsian group is properly discontinuous in $\mathfrak{A}_{(n,m)}$.

Now let us consider a general fuchsian group $\mathfrak{G}$, and put

$$\theta_k(Z)=\sum_{\sigma\in\mathfrak{G}} |U_3Z+U_4|^{-k(n+m)}, \qquad (k\geqq 2) \tag{3}$$

where $\sigma(Z)=(U_1Z+U_2)(U_3Z+U_4)^{-1}$, that is, a displacement σ is induced by $U=\begin{pmatrix} U_1 & U_2 \\ U_3 & U_4 \end{pmatrix}\in\Gamma_{(n,m)}$.

Theorem: The series $\theta_k(Z)$ thus defined is absolutely and uniformly convergent in the neighbourhood of any point in $\mathfrak{A}_{(n,m)}$.

For the proof of this theorem we have only to calculate the euclidean volume $v(\sigma\mathfrak{K})$ of the set $\sigma\mathfrak{K}$, where $\sigma\in\mathfrak{G}$ and $\mathfrak{K}$ is the set of all points $Z=(z_{ik})$ such that $|z_{ik}-z^0_{ik}|<r$ for a fixed point $Z_0=(z^0_{ik})$. The volume is given by

$$v(\sigma\mathfrak{K})=\int_{\mathfrak{K}} IdZ^*, \quad dZ^*=\prod_{\alpha=1}^{n}\prod_{\beta=1}^{m} dx_{\alpha\beta}dy_{\alpha\beta}, \quad z_{\alpha\beta}=x_{\alpha\beta}+iy_{\alpha\beta},$$

where I means the absolute value of the Jacobian $\frac{\partial\sigma(Z)}{\partial Z}$ for the displacement σ. But we have here $I=\|U_3Z+U_4\|^{-2(n+m)}$ and consequently $v(\sigma\mathfrak{K})\geqq(\pi r^2)^{nm}\|U_3Z_0+U_4\|^{-2(n+m)}$

The second method of proof given in S. II is also applicable to our case. For this purpose we introduce a non-euclidean metric in the space $\mathfrak{A}_{(n,m)}$ by defining a line element as $ds^2=Sp[(E^{(m)}-\bar{Z}'Z)^{-1}(\overline{dZ})'(E^{(n)}-Z\bar{Z}')^{-1}dZ]$. Then the volume element dv is given by $dv=|E^{(m)}-\bar{Z}'Z|^{-(n+m)}dZ^*$. As for zetafuchsian functions we obtain an analogous theorem as in S. II.

§ 2. *Lemmas on matrices.*

Lemma 1: If A is a matrix of the type (n, m), then there exist two unitary matrices U (of order n) and V (of order m) such that

$$UAV=\begin{pmatrix} a_1 & & & 0 \\ & \ddots & & \\ & & a_r & \\ 0 & & & \ddots \end{pmatrix}, \qquad (a_i\geqq 0)^{1)}.$$

If A is in particular a symmetrical matrix, then Lemma 1 can be stated more precisely.

Lemma 2: If A is a symmetrical matrix of order n, then there

1) This is known. Cf. J. von Neumann, Trans. Amer. Math. Soc. **36**, 445–492. Let us find eigenvectors of $\bar{A}'A$: $\bar{A}'A\mathfrak{x}_i=\lambda_i\mathfrak{x}_i$, $\bar{\mathfrak{x}}_i'\mathfrak{x}_j=\delta_{ij}$. For positive $\lambda_i>0$ we put $\mathfrak{y}_i=\frac{1}{\sqrt{\lambda_i}}A\mathfrak{x}_i$, and then construct a complete orthonormal system $\mathfrak{z}_1,\dots,\mathfrak{z}_n$ which includes these $\mathfrak{y}_i$. Then we have $\bar{\mathfrak{z}}_i'A\mathfrak{x}_j=\sqrt{\lambda_i}\,\delta_{ij}$ or 0. This proves Lemma 1.

exists a unitary matrix U so that $U'AU$ is a real (non negative) diagonal matrix.

Proof. First we will show that the equation

$$A\mathfrak{z}=\lambda\bar{\mathfrak{z}} \tag{4}$$

has, for a suitable real number λ, a solution vector $\mathfrak{z}$ (of dimension n). For this purpose let us put $A=B+iC$ and $\mathfrak{z}=\mathfrak{x}+i\mathfrak{y}$, where B, C or $\mathfrak{x}$, $\mathfrak{y}$ are respectively real symmetrical matrices or real vectors. Then the equation (4) can be written as follows:

$$\begin{cases} (\lambda E^{(n)}-B)\mathfrak{x}+C\mathfrak{y}=0 \\ C\mathfrak{x}+(\lambda E^{(n)}+B)\mathfrak{y}=0\,. \end{cases} \tag{5}$$

Since $K=\begin{pmatrix} B & -C \\ -C & -B \end{pmatrix}$ is a real symmetrical matrix, the characteristic equation of K has only real roots. If we denote one of these roots by α_1, the equation (5), and consequently (4) has, for $\lambda=\alpha_1$, a non trivial solution $\mathfrak{z}_1$ with the property $\mathfrak{z}_1'\bar{\mathfrak{z}}_1=1$. For this vector the relation $\mathfrak{z}_1'A\mathfrak{z}_1=\alpha_1$ holds. If $\mathfrak{z}'\bar{\mathfrak{z}}_1=0$ for another vector $\mathfrak{z}$, then $\mathfrak{z}'A\mathfrak{z}_1=\mathfrak{z}_1'A\mathfrak{z}=0$. Hence we can prove this lemma by proceeding analogously as in the case of hermitian matrices.

Remark. If we restrict our consideration to the points in $\mathfrak{A}_{(n,n)}$, which are represented by symmetrical matrices, we obtain the space studied by Prof. Sugawara. In this space it is seen from Lemma 2 that there exists a displacement which carries the given points A and B into 0 and a diagonal matrix. This is a theorem obtained by G. Fubini in his recent paper[1].

§ 3. *The distance in the space* $\mathfrak{A}_{(n,m)}$. For any two points Z_1 and Z_2 in $\mathfrak{A}_{(n,m)}$ we define

$$D(Z_1, Z_2)=E^{(m)}-(E^{(m)}-\bar{Z}_1'Z_2)^{-1}(E^{(m)}-\bar{Z}_1'Z_1)\,(E^{(m)}-\bar{Z}_2'Z_1)^{-1}(E^{(m)}-\bar{Z}_2'Z_2)\,.$$

If $\sigma\in\mathfrak{B}_{(n,m)}$, then $D(Z_1,Z_2)$ and $D\big(\sigma(Z_1),\sigma(Z_2)\big)$ are equivalent. Therefore the characteristic roots of $D(Z_1,Z_2)$ are invariant under the displacements of $\mathfrak{B}_{(n,m)}$. We denote the non negative quadratic roots of these characteristic roots by $d_1,\dots,d_m$[2], and put

$$\rho(Z_1, Z_2)=\frac{1}{2}\left[\left(\log\frac{1+d_1}{1-d_1}\right)^2+\cdots+\left(\log\frac{1+d_m}{1-d_m}\right)^2\right]^{\frac{1}{2}}, \tag{a}$$

$$\rho^*(Z_1, Z_2)=\frac{1}{2}\log\frac{1+d}{1-d}\,,\qquad d=\max_{1\leq i\leq m} d_i\,. \tag{b}$$

Then ρ and ρ^* are both invariant metrics in $\mathfrak{A}_{(n,m)}$.

The case (*a*). It is shown that in the non-euclidean space $\mathfrak{A}_{(n,m)}$ with

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1) G. Fubini, Proc. Nat. Acad. Sci. U. S. A. **26**, 700–708.

2) The characteristic roots of $D(Z_1,Z_2)$ are all non negative real numbers less than 1.

$ds^2 = Sp[(E^{(m)} - \bar{Z}'Z)^{-1}(d\bar{Z}')(E^{(n)} - Z\bar{Z}')^{-1}dZ]$ (Cf. § 1)[1] the geodesics are given by

$$\sigma Z(t) = \begin{pmatrix} \frac{\lambda_1^t - \lambda_1^{-t}}{\lambda_1^t + \lambda_1^{-t}} & & 0 \\ & \ddots & \\ 0 & & \frac{\lambda_m^t - \lambda_m^{-t}}{\lambda_m^t + \lambda_m^{-t}} \\ 0 & \cdots\cdots & 0 \end{pmatrix} \qquad (n \geqq m),$$

where $\sigma \in \mathfrak{B}_{(n,m)}$, $\lambda_i > 0$, t is a real variable[2]. The distance from Z_1 to Z_2 along the geodesic is just $\rho(Z_1, Z_2)$.

The case (*b*). We have only to examine the triangle relation

$$\rho^*(A, C) + \rho^*(C, B) \geqq \rho^*(A, B). \tag{6}$$

We shall prove (6) in the case $n = m$; if $n > m$ (or $n < m$), the space $\mathfrak{A}_{(n,m)}$ can be isometrically embedded in $\mathfrak{A}_{(n,n)}$ (or $\mathfrak{A}_{(m,m)}$). By Lemma 1 in § 2 we can assume without loss of generality that A is a diagonal matrix $\begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{pmatrix}$ $(1 > a_i \geqq 0)$, and C is the null matrix. If we denote the norm of a matrix T by $n(T)$, then we have $\rho^*(A, 0) = \frac{1}{2} \log \frac{1 + n(A)}{1 - n(A)}$,

$$\rho^*(B, 0) = \frac{1}{2} \log \frac{1 + n(B)}{1 - n(B)} \quad \text{and} \quad \rho^*(A, B) = \frac{1}{2} \log \frac{1 + n(K)}{1 - n(K)},$$

where $\qquad K = N^{-1}(B - A)(E - \bar{A}'B)^{-1}M, \quad E - A\bar{A}' = N\bar{N}',$

$$E - \bar{A}'A = M\bar{M}'.$$

Hence the relation (6) is reduced to

$$\frac{1 + n(A)}{1 - n(A)} \cdot \frac{1 + n(B)}{1 - n(B)} \geqq \frac{1 + n(K)}{1 - n(K)}. \tag{7}$$

But, for the proof of (7), it is sufficient to show

$$n(K) \leqq \frac{n(A) + n(B)}{1 + n(A) \cdot n(B)}. \tag{8}$$

From the form of A we know that N and M can be chosen as follows:

$$N = M = \begin{pmatrix} \sqrt{1 - a_1^2} & & 0 \\ & \ddots & \\ 0 & & \sqrt{1 - a_n^2} \end{pmatrix}.$$

By definition we get

$$n(K) = \underset{|\mathfrak{x}|=1}{\text{l. u. b.}} \| M^{-1}(B - A)(E - AB)^{-1}M\mathfrak{x} \| = \underset{|\mathfrak{x}|=1}{\text{l. u. b.}} \frac{\| M^{-1}(B - A)\mathfrak{x} \|}{\| M^{-1}(E - AB)\mathfrak{x} \|},$$

where $\mathfrak{x}$ is an n-dimensional vector and $\| \mathfrak{z} \|$ denotes the length of a

1) In the case $m = 1$ (hyperfuchsian groups) it is easily seen that

$$ds^2 = [(1 - \Sigma |z_i|^2)(\Sigma dz_i \overline{dz_i}) + |\Sigma z_i \overline{dz_i}|^2](1 - \Sigma |z_i|^2)^{-2}, \quad \Sigma = \sum_{i=1}^{n}, \quad Z = \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix}.$$

2) G. Fubini, Proc. Nat. Acad. Sci. U. S. A. **26**, 695-700.

vector $\mathfrak{z}$. If $\|\mathfrak{x}\|=1$, then $\|B\mathfrak{x}\|\leqq n(B)$. Hence for the proof of (8) it suffices to show

$$(9)\qquad \frac{\|M^{-1}\mathfrak{y}-M^{-1}A\mathfrak{x}\|}{\|M^{-1}\mathfrak{x}-M^{-1}A\mathfrak{y}\|}\leqq\frac{\alpha+\beta}{1+\alpha\beta},\quad \text{for}\quad \|\mathfrak{x}\|=1,\quad \|\mathfrak{y}\|=\beta,$$

where $\alpha=n(A)$, $\beta=n(B)$ and $\mathfrak{y}$ is a vector.

Putting $\varphi=\|M^{-1}\mathfrak{y}-M^{-1}A\mathfrak{x}\|$, $\psi=\|M^{-1}\mathfrak{x}-M^{-1}A\mathfrak{y}\|$,

$$\mathfrak{x}=\begin{pmatrix}x_1\\ \vdots\\ x_n\end{pmatrix},\quad \mathfrak{y}=\begin{pmatrix}y_1\\ \vdots\\ y_n\end{pmatrix},$$

we get

$$\varphi^2=\sum_{i=1}^{n}\frac{|x_i|^2+|y_i|^2}{1-\alpha_i^2}-1-2\Re\Big(\sum_{i=1}^{n}\frac{\alpha_i x_i\bar{y}_i}{1-\alpha_i^2}\Big)$$

$$\psi^2=\sum_{i=1}^{n}\frac{|x_i|^2+|y_i|^2}{1-\alpha_i^2}-\beta^2-2\Re\Big(\sum_{i=1}^{n}\frac{\alpha_i x_i\bar{y}_i}{1-\alpha_i^2}\Big).$$

Since $\dfrac{\varphi^2+\varepsilon_1+\varepsilon_2}{\psi^2+\varepsilon_1+\varepsilon_2}=\Big(\dfrac{\alpha+\beta}{1+\alpha\beta}\Big)^2<1$ and $\varepsilon_1\geqq 0$[1], $\varepsilon_2\geqq 0$[2], where

$$\varepsilon_1=\sum\frac{|x_i|^2+|y_i|^2}{1-\alpha^2}-\sum\frac{|x_i|^2+|y_i|^2}{1-\alpha_i^2},\quad \varepsilon_2=\frac{2\alpha\beta}{1-\alpha^2}+2\Re\Big(\sum\frac{\alpha_i x_i\bar{y}_i}{1-\alpha_i^2}\Big),$$

we have

$$\frac{\varphi^2}{\psi^2}\leqq\frac{\varphi^2+\varepsilon_1+\varepsilon_2}{\psi^2+\varepsilon_1+\varepsilon_2}=\Big(\frac{\alpha+\beta}{1+\alpha\beta}\Big)^2$$

and consequently inequality (6) is proved.

Remark. The above proof is valid for the space studied by Prof. Sugawara (Cf. S. III). For this case he has given a simple proof with an infinitesimal method.

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1) Because $\alpha=\text{Max}\,(\alpha_i)$.

2) This follows from Cauchy-Schwarz's inequality.