70. On the Conformal Arc Length.

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Let us consider a curve C in a conformally connected manifold C_n whose conformal connection is defined by the formulae

$$(1)\qquad \begin{cases} dA_0 = du^i A_i, \\ dA_j = \Pi^0_{jk} du^k A_0 + \Pi^i_{jk} du^k A_i + \Pi^\infty_{jk} du^k A_\infty, \\ dA_\infty = \Pi^i_{\infty k} du^k A_i, \end{cases}$$

where

$$(2)\qquad \Pi^i_{\infty k} = g^{ij}\Pi^0_{jk}, \quad \Pi^\infty_{jk} = g_{jk} \quad \text{and} \quad g^{ij}g_{jk} = \delta^i_k. \qquad (i, j, k, \dots = 1, 2, 3, \dots, n)$$

Defining two parameters s and t on the curve by the equations

$$(3)\qquad g_{jk}\frac{du^j}{ds}\frac{du^k}{ds} = 1$$

and

$$(4)\qquad \{t, s\} = \frac{1}{2} g_{jk}\frac{\delta^2 u^j}{\delta s^2}\frac{\delta^2 u^k}{\delta s^2} - \Pi^0_{jk}\frac{du^j}{ds}\frac{du^k}{ds}$$

respectively, where

$$(5)\qquad \{t, s\} = \frac{d^3 t}{ds^3}\Big/\frac{dt}{ds} - \frac{3}{2}\left(\frac{d^2 t}{ds^2}\Big/\frac{dt}{ds}\right)^2$$

and

$$(6)\qquad \frac{\delta^2 u^i}{\delta s^2} = \frac{d^2 u^i}{ds^2} + \Pi^i_{jk}\frac{du^j}{ds}\frac{du^k}{ds},$$

we find the following Frenet formulae[1)]

$$(7)\qquad \begin{cases} \underset{(0)}{S} = \dfrac{dt}{ds}A_0, \quad \underset{(1)}{S} = \dfrac{d}{dt}\underset{(0)}{S}, \quad \underset{(2)}{S} = \dfrac{d}{dt}\underset{(1)}{S}, \\ \dfrac{d}{dt}\underset{(2)}{S} = -\overset{3}{\varkappa}\underset{(3)}{S}, \\ \dfrac{d}{dt}\underset{(3)}{S} = -\overset{3}{\varkappa}\underset{(0)}{S} + \overset{4}{\varkappa}\underset{(4)}{S}, \\ \dfrac{d}{dt}\underset{(4)}{S} = -\overset{4}{\varkappa}\underset{(3)}{S} + \overset{5}{\varkappa}\underset{(5)}{S}, \\ \dots\dots\dots\dots\dots\dots \\ \dfrac{d}{dt}\underset{(\infty)}{S} = -\overset{\infty}{\varkappa}\underset{(n)}{S}, \end{cases}$$

1) See, K. Yano, Sur la théorie des espaces à connexion conforme. Journal of the Faculty of Science, Tokyo Imperial University, Sec. I, Vol. IV, Part 1 (1939), 1–59.

where $\underset{(0)}{S}$ and $\underset{(2)}{S}$ are two point-spheres and $\underset{(1)}{S}, \underset{(3)}{S}, \ldots\ldots, \underset{(n)}{S}, \underset{(\infty)}{S}$, n mutually orthogonal unit spheres all passing through the points $\underset{(0)}{S}$ and $\underset{(2)}{S}$.

The parameter t being defined by a Schwarzian differential equation, we shall call it projective parameter. The parameter t' defined by

$$t'=\frac{at+b}{ct+d} \tag{8}$$

is also a projective parameter. If we effect a transformation of the projective parameter t of the form (8), the curvature $\overset{3}{\varkappa}, \overset{4}{\varkappa}, \ldots, \overset{\infty}{\varkappa}$ appearing in the Frenet formulae (7) will be respectively transformed into $\overset{3}{\varkappa}{}', \overset{4}{\varkappa}{}', \ldots, \overset{\infty}{\varkappa}{}'$, where

$$\overset{3}{\varkappa}{}'=\left(\frac{dt}{dt'}\right)^{2}\overset{3}{\varkappa}, \quad \overset{4}{\varkappa}{}'=\left(\frac{dt}{dt'}\right)^{4}\overset{4}{\varkappa}, \quad \ldots\ldots, \quad \overset{\infty}{\varkappa}{}'=\left(\frac{dt}{dt'}\right)^{\infty}\overset{\infty}{\varkappa}\text{[1]}. \tag{9}$$

Hence, we can see that the differential

$$d\sigma=(\overset{3}{\varkappa})^{\frac{1}{2}}dt \tag{10}$$

is a conformal invariant, it is the conformal invariant of the least degree.

We shall call σ the conformal arc length of the curve. The conformal arc length does not exist for a generalized circle[2].

The conformal arc length σ being thus defined, the point

$$\underset{(0)}{A}=\frac{d\sigma}{dt}\underset{(0)}{S}=\frac{d\sigma}{ds}A_0 \tag{11}$$

is a not only geometrically but also analytically invariant point.

Differentiating the point $\underset{(0)}{A}$ with respect to σ, we have

$$\underset{(1)}{A}=\frac{d}{d\sigma}\underset{(0)}{A}=\frac{\dfrac{d^2\sigma}{dt^2}}{\dfrac{d\sigma}{dt}}\underset{(0)}{S}+\underset{(1)}{S}, \tag{12}$$

hence, we can see that

$$\underset{(0)}{A}\underset{(0)}{A}=0, \quad \underset{(0)}{A}\underset{(1)}{A}=0, \quad \underset{(1)}{A}\underset{(1)}{A}=1, \tag{13}$$

that is, $\underset{(1)}{A}$ is an analytically invariant unit sphere passing through the point $\underset{(0)}{A}$. Differentiating the unit sphere $\underset{(1)}{A}$ with respect to σ, we obtain

$$\frac{d}{d\sigma}\underset{(1)}{A}=\frac{\dfrac{d^3\sigma}{dt^3}}{\left(\dfrac{d\sigma}{dt}\right)^2}\underset{(0)}{S}-\frac{\left(\dfrac{d^2\sigma}{dt^2}\right)^2}{\left(\dfrac{d\sigma}{dt}\right)^3}\underset{(0)}{S}+\frac{\dfrac{d^2\sigma}{dt^2}}{\left(\dfrac{d\sigma}{dt}\right)^2}\underset{(1)}{S}+\frac{1}{\dfrac{d\sigma}{dt}}\underset{(2)}{S}$$

1) K. Yano and Y. Mutô, Sur la théorie des hypersurfaces dans un espace à connexion conforme. Japanese Journal of Mathematics, **17** (1941), 229–288.

2) K. Yano, loc. cit.

or

$$(14)\qquad \frac{d}{d\sigma}\underset{(1)}{A}=\frac{\{\sigma,t\}}{\left(\frac{d\sigma}{dt}\right)^2}\underset{(0)}{A}+\frac{1}{2}\frac{\left(\frac{d^2\sigma}{dt^2}\right)^2}{\left(\frac{d\sigma}{dt}\right)^3}\underset{(0)}{S}+\frac{\frac{d^2\sigma}{dt^2}}{\left(\frac{d\sigma}{dt}\right)^2}\underset{(1)}{S}+\frac{1}{\frac{d\sigma}{dt}}\underset{(2)}{S}.$$

Now, putting

$$(15)\qquad \lambda=\frac{\{\sigma,t\}}{\left(\frac{d\sigma}{dt}\right)^2}=-\{t,\sigma\}$$

and

$$(16)\qquad \underset{(2)}{A}=\frac{1}{2}\frac{\left(\frac{d^2\sigma}{dt^2}\right)^2}{\left(\frac{d\sigma}{dt}\right)^3}\underset{(0)}{S}+\frac{\frac{d^2\sigma}{dt^2}}{\left(\frac{d\sigma}{dt}\right)^2}\underset{(1)}{S}+\frac{1}{\frac{d\sigma}{dt}}\underset{(2)}{S},$$

we find from (14)

$$(17)\qquad \frac{d}{d\sigma}\underset{(1)}{A}=\lambda\underset{(0)}{A}+\underset{(2)}{A}.$$

The function λ being defined by the Schwarzian derivative $-\{t,\sigma\}$ where t is a projective parameter and σ is a conformal one, the equation (17) shows that the sphere $\underset{(2)}{A}$ defined by (16) is an analytically invariant one. Moreover, the equations (11), (12) and (16) shows that

$$\underset{(0)}{A}\underset{(0)}{A}=\underset{(2)}{A}\underset{(2)}{A}=0,\quad \underset{(0)}{A}\underset{(1)}{A}=\underset{(2)}{A}\underset{(1)}{A}=0,\quad \underset{(1)}{A}\underset{(1)}{A}=1,\quad \underset{(0)}{A}\underset{(2)}{A}=-1,$$

that is $\underset{(2)}{A}$ is a point on the sphere $\underset{(1)}{A}$ satisfying the relation $\underset{(0)}{A}\underset{(2)}{A}=-1$.

Substituting the relation

$$\frac{d\sigma}{dt}=(\overset{3}{\varkappa})^{\frac{1}{2}}$$

in (15), we can find the expression

$$(18)\qquad \lambda=\frac{1}{2}\left[\frac{1}{(\overset{3}{\varkappa})^2}\frac{d^2\overset{3}{\varkappa}}{dt^2}-\frac{5}{4}\frac{1}{(\overset{3}{\varkappa})^3}\left(\frac{d\overset{3}{\varkappa}}{dt}\right)^2\right],$$

which is a conformal invariant we have already found in a previous paper[1].

Now, differentiating $\underset{(2)}{A}$ along the curve with respect to σ, we obtain

$$\frac{d}{d\sigma}\underset{(2)}{A}=\left[\frac{\frac{d^2\sigma}{dt^2}\frac{d^3\sigma}{dt^3}}{\left(\frac{d\sigma}{dt}\right)^4}-\frac{3}{2}\frac{\left(\frac{d^2\sigma}{dt^2}\right)^3}{\left(\frac{d\sigma}{dt}\right)^5}\right]\underset{(0)}{S}+\left[\frac{\frac{d^3\sigma}{dt^3}}{\left(\frac{d\sigma}{dt}\right)^3}-\frac{3}{2}\frac{\left(\frac{d^2\sigma}{dt^3}\right)^2}{\left(\frac{d\sigma}{dt}\right)^4}\right]\underset{(1)}{S}-\frac{\overset{3}{\varkappa}}{\left(\frac{d\sigma}{dt}\right)^2}\underset{(3)}{S}$$

1) K. Yano and Y. Mutô. loc. cit.

$$=\frac{\{\sigma,t\}\dfrac{d^2\sigma}{dt^2}}{\left(\dfrac{d\sigma}{dt}\right)^3}\underset{(0)}{S}+\frac{\{\sigma,t\}}{\left(\dfrac{d\sigma}{dt}\right)^2}\underset{(1)}{S}-\frac{\overset{3}{\varkappa}}{\left(\dfrac{d\sigma}{dt}\right)^2}\underset{(3)}{S}$$

$$=\frac{\{\sigma,t\}}{\left(\dfrac{d\sigma}{dt}\right)^2}\left[\frac{\dfrac{d^2\sigma}{dt^2}}{\dfrac{d\sigma}{dt}}\underset{(0)}{S}+\underset{(1)}{S}\right]-\frac{\overset{3}{\varkappa}}{\left(\dfrac{d\sigma}{dt}\right)^2}\underset{(3)}{S}$$

from which

(19)
$$\frac{d}{d\sigma}\underset{(2)}{A}=\lambda\underset{(1)}{A}-\underset{(3)}{S}$$

in virtue of the relation

$$\lambda=\frac{\{\sigma,t\}}{\left(\dfrac{d\sigma}{dt}\right)^2},\quad \underset{(1)}{A}=\frac{\dfrac{d^2\sigma}{dt^2}}{\dfrac{d\sigma}{dt}}\underset{(0)}{S}+\underset{(1)}{S}\quad\text{and}\quad \frac{\overset{3}{\varkappa}}{\left(\dfrac{d\sigma}{dt}\right)^2}=1\,.$$

The equation (19) shows that the $\underset{(3)}{S}$ is an analytically invariant unit sphere, hence, if we put

(20)
$$\underset{(3)}{A}=-\underset{(3)}{S},$$

we have from (19)

(21)
$$\frac{d}{d\sigma}\underset{(2)}{A}=\lambda\underset{(1)}{A}+\underset{(3)}{A}\,.$$

It will be easily verified that the unit sphere $\underset{(3)}{A}$ passes through the two points $\underset{(0)}{A}$ and $\underset{(2)}{A}$ and is orthogonal to the unit sphere $\underset{(1)}{A}$.

Differentiating the unit sphere $\underset{(3)}{A}$ along the curve with respect to σ, we have

(22)
$$\frac{d}{d\sigma}\underset{(3)}{A}=-\frac{1}{\dfrac{d\sigma}{dt}}\left[-\overset{3}{\varkappa}\underset{(0)}{S}+\overset{4}{\varkappa}\underset{(4)}{S}\right],$$

consequently, if we put

(23)
$$\overset{4}{\lambda}=\frac{\overset{4}{\varkappa}}{\dfrac{d\sigma}{dt}}=\overset{4}{\varkappa}(\overset{3}{\varkappa})^{-\frac{1}{2}},\quad \underset{(4)}{A}=-\underset{(4)}{S},$$

we obtain from (22)

(24)
$$\frac{d}{d\sigma}\underset{(3)}{A}=\underset{(0)}{A}+\overset{4}{\lambda}\underset{(4)}{A}\,.$$

This equation tells us that the unit sphere $\underset{(4)}{A}$ is an analytically invariant sphere, $\overset{4}{\lambda}$ being a conformal invariant.

The sphere $\underset{(4)}{A}$ passes through the points $\underset{(0)}{A}$ and $\underset{(2)}{A}$ and is orthogonal to the spheres $\underset{(1)}{A}$ and $\underset{(3)}{A}$. Differentiating

$$\underset{(4)}{A} = -\underset{(4)}{S},$$

we obtain

$$\frac{d}{d\sigma}\underset{(4)}{A} = -\overset{4}{\lambda}\underset{(3)}{A} + \overset{5}{\lambda}\underset{(5)}{A}, \tag{25}$$

where

$$\overset{5}{\lambda} = \overset{5}{\varkappa}(\overset{3}{\varkappa})^{-\frac{1}{2}} \tag{26}$$

is a conformal invariant and

$$\underset{(5)}{A} = -\underset{(5)}{S} \tag{27}$$

is an analytically invariant unit sphere passing through the points $\underset{(0)}{A}$ and $\underset{(2)}{A}$ and is orthogonal to the spheres $\underset{(1)}{A}$, $\underset{(3)}{A}$ and $\underset{(4)}{A}$.

Continuing in this way, we obtain finally the following conformal Frenet formulae

$$(28)\quad \begin{cases} \dfrac{d}{d\sigma}\underset{(0)}{A} = \underset{(1)}{A}, \\ \dfrac{d}{d\sigma}\underset{(1)}{A} = \lambda\underset{(0)}{A} + \underset{(2)}{A}, \\ \dfrac{d}{d\sigma}\underset{(2)}{A} = \lambda\underset{(1)}{A} + \underset{(3)}{A}, \\ \dfrac{d}{d\sigma}\underset{(3)}{A} = \underset{(0)}{A} + \overset{4}{\lambda}\underset{(4)}{A}, \\ \dfrac{d}{d\sigma}\underset{(4)}{A} = -\overset{4}{\lambda}\underset{(3)}{A} + \overset{5}{\lambda}\underset{(5)}{A}, \\ \dots\dots\dots\dots\dots\dots \\ \dfrac{d}{d\sigma}\underset{(n)}{A} = -\overset{n}{\lambda}\underset{(n-1)}{A} + \overset{\infty}{\lambda}\underset{(\infty)}{A}, \\ \dfrac{d}{d\sigma}\underset{(\infty)}{A} = -\overset{\infty}{\lambda}\underset{(n)}{A}, \end{cases}$$

where $\underset{(0)}{A}$ and $\underset{(2)}{A}$ are two points and $\underset{(1)}{A}$, $\underset{(3)}{A}$, $\cdots$, $\underset{(\infty)}{A}$ are n mutually orthogonal unit spheres passing through the points $\underset{(0)}{A}$ and $\underset{(2)}{A}$, and

$$\lambda = -\{t, \sigma\}, \quad \overset{4}{\lambda} = \overset{4}{\varkappa}(\overset{3}{\varkappa})^{-\frac{1}{2}}, \quad \overset{5}{\lambda} = \overset{5}{\varkappa}(\overset{3}{\varkappa})^{-\frac{1}{2}}, \quad \dots\dots, \quad \overset{\infty}{\lambda} = \overset{\infty}{\varkappa}(\overset{3}{\varkappa})^{-\frac{1}{2}}. \tag{29}$$

The consequences of these conformal Frenet formulae will be developed in the forthcoming papers.