66. *On the Behaviour of a Meromorphic Function in the Neighbourhood of a Transcendental Singularity.*

By Yukinari TÔKI.

Mathematical Institute, Osaka Imperial University.

(Comm. by S. KAKEYA, M.I.A., Oct. 11, 1941.)

In this paper we shall prove the theorems of Beurling-Kunugui[1], Kunugui[2], and Iversen-Gross[3] using L. Ahlfors' principal theorem on covering surfaces[4].

Suppose that $f(z)$ is uniform, meromorphic in a connected domain D. Let z_0 be a point on the boundary Γ of D. We associate with z_0 three sets of values:

(1) The cluster set $S_{z_0}^{(D)}$. This is the set of all values α such that $\lim_{\nu\to\infty} f(z_\nu)=\alpha$ where z_ν $(\nu=1, 2, \dots)$ is a sequence of points tending to z_0 within D. It is obvious that $S_{z_0}^{(D)}$ is a closed set.

(2) The cluster set $S_{z_0}^{(\Gamma)}$. This is the product $\prod_{n=1}^{\infty} M_n$, where M_n denotes the closure of the sum $\sum_{0<|z'-z_0|<\frac{1}{n}} S_{z'}^{(D)}$, z' belonging to Γ. This set is also a closed set and $S_{z_0}^{(D)}$ includes $S_{z_0}^{(\Gamma)}$.

(3) The range of values $R_{z_0}^{(D)}$. A value α belongs to $R_{z_0}^{(D)}$ if, and only if, $f(z)$ takes the value α an infinity of times near z_0 inside D. It is obvious that $S_{z_0}^{(D)}$ includes $R_{z_0}^{(D)}$.

Suppose that $d(S_1, S_2)$ denotes the distance between a set S_1 and a set S_2, CS the complement of a set S with respect to the w-plane, $\bar{S}$ the closure of S, $B(S)$ the boundary of S, and $K(r)$ and $k(r)$ denote the circular disc $|z-z_0|<r$ and circumference $|z-z_0|=r$ respectively.

Lemma 1[5]. *Let $w=f(z)$ be uniform and meromorphic in a domain D, and z_0 be a point on the boundary Γ of D. Suppose that α is a value belonging to $S_{z_0}^{(D)}-S_{z_0}^{(\Gamma)}$ but not belonging to $R_{z_0}^{(D)}$. Then α is an asymptotic value of $f(z)$ at z_0 and the length of the image of its asymptotic path by $w=f(z)$ on the Riemann sphere is finite.*

Proof. We may assume that α is finite by rotating the Riemann sphere, if necessary. Since $\alpha \bar{\in} S_{z_0}^{(\Gamma)}$, $\alpha \bar{\in} R_{z_0}^{(D)}$, there exists a positive number r such that $\alpha \bar{\in} \overline{\sum_{0<|z'-z_0|\leq r} S_{z'}^{(D)}}$ where z' varies on Γ, and $f(z) \neq \alpha$ for $|z-z_0|\leqq r$ within D. Consequently there exist positive numbers

---

1) K. Kunugui: Sur un théorème de MM. Seidel-Beurling, Proc. **15** (1939), 27–32.

2) K. Kunugui: Sur un problème de M. A. Beurling, Proc. **16** (1940), 361–366.

3) K. Noshiro: On the theory of the cluster sets of analytic functions, Journ. Fac. Sc. Hokkaido Imp. University. Ser. I, vol. **6** (1938), pp. 230–231.

4) L. Ahlfors: Zur Theorie der Überlagerungsflächen, Acta Math., Bd. **65** (1935), or R. Nevanlinna: Eindeutige analytischen Funktionen, (1936), pp. 312–345.

5) Noshiro: loc. cit. theorem 1. p. 221. He proved that α is an asymtotic value of $f(z)$ at z_0 under the same hypothesis.

ρ_1 and ρ_2, such as $\rho_1=d(a, \overline{\sum_{0<|z'-z_0|\leq r} S_{z'}^{(D)}})$, $\rho_2=d\left(a, \overline{\sum_{z\in k(r)\cdot D} f(z)}\right)$. We can find a positive number ρ such that $\rho<\min\left(\frac{\rho_1}{2}, \frac{\rho_2}{2}\right)$. Then the condition (A) is satisfied.

(A): The sets $\overline{\sum_{0<|z'-z_0|\leq r} S_{z'}^{(D)}}$ and $\overline{\sum_{z\in k(r)\cdot D} f(z)}$ both lie outside the circle (c): $|w-a|<\rho$.

Remember that a belongs to $S_{z_0}^{(D)}$. Then there is a sequence $\{z_\nu\}$ $(\nu=1, 2, \dots)$ of points within D such that $z_\nu\to z_0$ and $f(z_\nu)\to a$. Accordingly we may find a point ζ_0 in $K(r)\cdot D$, whose image $w_0=f(\zeta_0)$ lies within the circle (c), and denote by e_{ζ_0} an inverse element of $w=f(z)$ corresponding to $z=\zeta_0$.

Continuing analytically the element e_{ζ_0} within the circle (c) with algebraic characters, we obtain a branch $z=\varphi_{\Delta_0}(w)$ of the inverse function of $w=f(z)$; the set of values taken by $z=\varphi_{\Delta_0}(w)$ is a connected domain Δ_0. By (A) Δ_0 satisfies the condition (B).

(B): $B(\Delta_0)$ has a point z_0 and $f(z)$ is regular in the closed domain $\bar{\Delta}_0$ except at z_0 and the image of $B(\Delta_0)$ except at z_0 is arcs on the circumference $|w-a|=\rho$.

Consequently we can find an angle $\theta\ (0<\theta<2\pi)$ such that the length of the image of a curve Γ' by $w=f(z)$, where Γ' is a part of $B(\Delta_0)$ exclusive of z_0, is $\rho\theta$ on the circumference $|w-a|=\rho$.

If we can find a point ζ on Γ' with its inverse element e_ζ such that we can continue analytically the element e_ζ along a radius of the circle (c) except at $w=a$ with no algebraic character, we obtain a simple analytical curve Π having two end points ζ and z_0 such that the length of image of Π by $w=f(z)$ is ρ, and lemma 1 is proved.

If such a path Π never exists, there exists a positive number ρ_0 $(\rho_0<\rho)$ such that the projection of Ω to the w-plane lies outside the circle (c): $|w-a|<\rho_0$, where Ω is the sum of all open segments on the radii of the circle (c) along which except at the end points we can continue analytically (with no algebraic character) each inverse element e_z corresponding to z on Γ'. Let Δ' is the image of the set Ω by $z=\varphi_{\Delta_0}(w)$ whose boundary includes Γ', it is obvious by (B) that $B(\Delta')$ includes z_0. We can find a positive number r_0 such that $k(r_0)$ is cut off by Δ'. Let $L(r)$ be the length of the image of the cross-cuts $k(r)\cdot\Delta'$ $(0<r<r_0)$ by the function $w=f(z)$, it is longer than $\rho_0\theta$. Let $A(r)$ denote the area of the image of the open set $\Delta'(r)$ by the function $w=f(z)$ where $\Delta'(r)$ means the subset of Δ' lying between $k(r_0)$ and $k(r)$. Then the well-known inequality $\log\frac{r_0}{r}\leqq -2\pi\int_r^{r_0}\frac{dA(r)}{L^2(r)}$ is satisfied. Also $L(r)\geqq\rho_0\theta>0$,

hence
$$\log\frac{r_0}{r}\leqq\frac{2\pi}{\rho_0\theta}A(r),$$

whence it follows that
$$\lim_{r\to 0} A(r)=\infty.$$

On the other hand $A(r)<\frac{1}{2}\rho^2\theta<\infty$. Thus we arrive at a contradiction.

Lemma 2. *Let $w=f(z)$ be uniform and meromorphic in a domain D and z_0 be a point on the boundary Γ of D. Suppose that a domain Ω belongs to the open set $CS_{z_0}^{(\Gamma)}$. If a value α belongs to $S_{z_0}^{(D)}$ and to Ω, and another value β belongs to Ω but not to $R_{z_0}^{(D)}$, then Ω has at least an asymptotic value.*

Proof. i) Suppose that α does not belong to $R_{z_0}^{(D)}$. Then α is an asymptotic value by the Lemma 1 or Noshiro's theorem[1].

ii) Suppose that α belongs to $R_{z_0}^{(D)}$. Now let $z=\varphi(w)$ be the inverse function of $w=f(z)$, and $z=\varphi(w)$ may contain at most an enumerable infinity of ramified elements. Consequently a path L (consisting of a finite number of segments) may be drawn in Ω, running from $w=\alpha$ to $w=\beta$, such that there exists no centre of ramified element of the invers function $z=\varphi(w)$ on L excluding the end-points $w=\alpha$ and $w=\beta$.

Let r is a sufficiently small positive number, then $f(z)\neq\beta$ for $|z-z_0|<r$ within D and $d(L, \overline{\sum_{0<|z'-z_0|\leq r} S_{z'}^{(D)}})>0$ where z' varies on Γ, for $\beta\,\bar{\in}\, R_{z_0}^{(D)}$ and $\Omega\cdot S_{z_0}^{(\Gamma)}=0$. Let $z_1, z_2, \dots, z_\nu, \dots$ denote all the α-points within $K(r)\cdot D$ and $e_{z_1}, e_{z_2}, \dots, e_{z_\nu}, \dots$ all the corresponding inverse elements with centres at $w=\alpha$.

We obtain an infinite number of paths $\sum_{\nu=1}^{\infty}\Pi_\nu$ on the z-plane, where Π_ν is the image corresponding to the path of the analytic continuation of the element e_{z_1} by $z=\varphi(w)$ along L with no algebraic character.

Since $d(L, \sum_{0<|z'-z_0|\leq r} S_{z'}^{(D)})$ is positive, $k(r)$ may cross at most a finite number of paths $\sum_n \Pi_{\nu_n}$. Since there exists no β-point within $K(r)\cdot D$, the paths $\sum_{\nu=1}^{\infty}\Pi_\nu$ are the asymptotic paths excepting the finite number of paths $\sum_n \Pi_{\nu_n}$.

Thus Lemma 2 has proved.

Theorem. *Let $w=f(z)$ be uniform and meromorphic in the domain D whose boundary is denoted by Γ. Suppose that z_0 is a non-isolated boundary point and Ω is an arbitrary domain including at least one value of $S_{z_0}^{(D)}$ and belongs to the open set $CS_{z_0}^{(\Gamma)}$. Then $R_{z_0}^{(D)}$ includes every value of Ω except at most two values.*

Proof[2]. We will prove that if Ω includes three distinct values α_1, α_2, α_3 not belonging to $R_{z_0}^{(D)}$, then we arrive at a contradiction. By Lemma 2 there exists at least one asymptotic value α_0 belonging to Ω.

Let its asymptotic path be denoted by Π, and let $\alpha_0\neq\alpha_1$. Then there exists a simply connected domain Ω' included Ω and satisfying following conditions.

i) $\alpha_0, \alpha_2, \alpha_3\in\Omega'$.

ii) $\alpha_1\in B(\Omega')$ and $B(\Omega')$ consists of a finite number of segments.

iii) $d\big(B(\Omega), B(\Omega')\big)>0$, $d\big(\alpha_0, B(\Omega')\big)>\varepsilon$ for a sufficiently small positive number ε, where d is a spherical distance.

---

1) Noshiro: loc. cit.

2) For the method in our proof we owe much to Kunugui. Cf. Kunugui: loc. cit.

Then

iv) $d(\Omega', \overline{\sum_{0<|z'-z_0|\leq r_0} S_{z'}^{(D)}})>0\,, \ z'\in\Gamma\,,$ and $f(z)\neq a_2, a_3\,, \ z\in K(r_0)\cdot D$ for a sufficiently small positive number r_0.

Let p be a point sufficently near z_0 on Π such that the image of the path Π from p on by $w=f(z)$ lies in Ω'. Continuing analytically the inverse element e_p of p within Ω' with the algebraic characters, we obtain a branch $z=\varphi_\Delta(w)$ of inverse function of $w=f(z)$; the set of values taken by $z=\varphi_\Delta(w)$ in Ω' is a connected domaim Δ and its boundary $B(\Delta)$ includes z_0.

From iv) and $a_1\in B(\Omega')$, it is obvious that the image of $B(\Delta)$ within $K(r_0)$ except at z_0 by $w=f(z)$ lies on $B(\Omega')$, and at most a finite number of boundaries may cross $k(r_0)$, and $B(\Delta)$ within $K(r_0)$ has no closed contour on which z_0 does not lie. Remember that z_0 is the non-isolated boundary point, it is evident that there exists a continuous boundary curve Γ' of Δ on which z_0 lies and $K(r_0)\cdot\Delta$ is a finite number of simply connected domains. Accordingly the set of Δ lying between $k(r_0)$ and $k(r)$ $(r<r_0)$ is the sum of simply connected domains denoted by $\Delta_i(r)$ $(i=1, 2, \dots, n)$.

Now let us introduce some notations to apply Ahlfors' theorem. Let Ω_0, which coincides with Ω' excluding a_2 and a_2, be the basic surface (*Grundfläche*), $\Delta_i(r)$ $(i=1, 2, \dots, n)$ covering surfaces (*Überlagerungsfläche*) of Ω_0, and $L_i(r)$ the length (on the Riemann sphere) of the boundary of $\Delta_i(r)$ relative to Ω_0, and $S_i(r)$ the average number of sheets (*mittlere Blätteranzahl*) of $\Delta_i(r)$ (on the Riemann sphere).

Apply Ahlfors' principal theorem to each covering surface $\Delta_i(r)$ (*Euler characteristic* -1) of basic surface Ω_0 (*Euler characteristic* $+1$).

$$0\geq S_i(r)-hL_i(r)\,,$$

where h is a constant only depending on Ω_0.

Hence $$0\geq S(r)-hL(r)\,,$$

where $$S(r)=\sum_{i=1}^{n} S_i(r)\,, \quad L(r)=\sum_{=1}^{n} L_i(r)\,, \quad \text{then}$$

v) $$0\geq 1-h\frac{L(r)}{S(r)}\,.$$

On the other hand $k(r)$ necessarily crosses Π and Γ', so $L(r)>\varepsilon>0$ for sufficiently small positive number r. Accordingly $\lim_{r\to 0} S(r)=\infty$ as the proof of Lemma 1, from which we can easily prove that there exists a sequence r_ν $(\nu=1, 2, \dots)$ tending to zero, and $\lim_{\nu\to\infty}\frac{L(r_\nu)}{S(r_\nu)}=0$[1]. Therefore v) becomes

$$0\geq 1\,.$$

Thus we arrive at a contradiction. (Q. E. D.)

This theorem shows that any one of components of the complementary set of $S_{z_0}^{(\Gamma)}$ with respect to the w-plane is entirely included in

1) R. Nevanlinna: loc. cit. pp. 340–341.

$S_{z_0}^{(D)}$ or never contains any point lying in $S_{z_0}^{(D)}$. Accordingly $S_{z_0}^{(D)}-S_{z_0}^{(\Gamma)}$ is an open set. Then we have at once next two theorems.

Theorem. (Beuling-Kunugui.) *Let D be an arbitrary connected domain and z_0 be a non-isolated boundary point. If $w=f(z)$ is uniform and holomorphic and bounded in the neighbourhood of z_0 within D, then it follows*

$$\overline{\lim_{z\to z_0}}\,|f(z)|=\overline{\lim_{z'\to z_0}}\left(\overline{\lim_{\substack{z\to z'\\ z'\neq z_0}}}\,|f(z)|\right),$$

denoting any boundary point by z'.

Theorem. (Kunugui). *Let $f(z)$ be uniform and meromorphic in a domain D whose boundary is denoted by Γ, and z_0 be a non-isolated point. If Ω is an arbitary domain included in $S_{z_0}^{(D)}-S_{z_0}^{(\Gamma)}$, then $R_{z_0}^{(D)}$ includes every value of Ω except at most two values.*

Now we shall prove Iversen-Gross' theorem extended by K. Noshiro.

Theorem. (Iversen-Gross-Noshiro.) *Let $w=f(z)$ be uniform and meromorphic in a simply connected domain D whose boundary is denoted by Γ, and z_0 is a simply accessible boundary point*[1]. *Then $R_{z_0}^{(D)}$ includes every value of $S_{z_0}^{(D)}-S_{z_0}^{(\Gamma)}$ except at most two values, and, if there are two such exceptional values, then $R_{z_0}^{(D)}$ must coincide with the w-plane excluding these two points.*

Proof. We will prove that if there exist three distinct points a_1, a_2, a_3 not belonging to $R_{z_0}^{(D)}$ and $a_1, a_2 \in S_{z_0}^{(D)}-S_{z_0}^{(\Gamma)}$, we arrive at a contradiction. By Lemma 1 a_1 and a_2 are asymptotic values at z_0 and the length of images of their asymptotic paths Π_1 and Π_2 are finite. There exists a positive number r_0 such that $k(r_0)$ necessarily cross Π_1 and Π_2. Let Δ denote a simply connected domain surrounded by Π_1, Π_2, and $k(r_0)$. The set of Δ lying between $k(r_0)$ and $k(r)$ $(r<r_0)$ is the sum of simply connected domains $\Delta_i(r)$ $(i=1, 2, \dots, n)$. Let the w-plane excluding three points: a_1, a_2, a_3, be the basic surface and $\Delta_i(r)$ be the covering surface.

Apply Ahlfors' principal theorem to each $\Delta_i(r)$. We shall arrive at a contradiction as the proof of previous theorem using that the length of the images (on the Riemann sphere) of Π_1 and Π_2 are finite.

1) We say that z_0 is the simply accessible boundary point of a domain D, when z_0 is accessible and any two paths converging to z_0 are necessarily connected by a curve inside $K(r)\cdot D$ where r is an arbitary small positive number.