90. On the Behaviour of an Inverse Function of a Meromorphic Function at its Transcendental Singular Point.

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Let $w=f(z)$ be a meromorphic function for $|z|<\infty$ with a transcendental singularity at $z=\infty$ and its inverse function $z=\varphi(w)$ have a transcendental singularity ω with $w=0$ as its projection on the w-plane. Denote $\varDelta$ the set of values taken by $z=\varphi(w)$ which defines the ρ-neighbourhood of an accessible boundary point ω of the Riemann surface F of $z=\varphi(w)$. $\varDelta$ is a domain on the z-plane bounded, in general, by an enumerable infinity of analytic curves. $|w|=|f(z)|<\rho$ in $\varDelta$ and $|w|=\rho$ on the boundary of $\varDelta$. Let z_0 be a point in $\varDelta$. The common part of $\varDelta$ and $|z-z_0|<r$ consists of a certain number of connected domains. Let $\varDelta_r$ be one of such domains which contains z_0. The boundary of $\varDelta_r$ consists of curves of three types: $\{a_i^{(r)}\}$, $\{b_i^{(r)}\}$, $\{c_i^{(r)}\}$, where $a_1^{(r)}, a_2^{(r)}, \dots, a_{n_r}^{(r)}$ are circular arcs on $|z-z_0|=r$, $b_1^{(r)}$, $b_2^{(r)}, \dots, b_{m_r}^{(r)}$ are the parts of the boundary of $\varDelta$ which meet $|z-z_0|=r$, and $c_1^{(r)}, c_2^{(r)} \dots, c_{p_r}^{(r)}$ are the closed curves which are the boundaries of holes in $\varDelta_r$. We put

$$\Lambda(r)=p_r=\text{number of holes in } \varDelta_r\,. \tag{1}$$

Let F_r correspond to $\varDelta_r$ on F and $A(r)$ be the area of F_r and put

$$S(r)=\frac{A(r)}{\pi\rho^2}\,. \tag{2}$$

$A(r)$ is an increasing function of r and is continuous except at most an enemerable infinity of points $\{a_i\}$, where $A(a_i-0)=A(a_i)$. Let $A_i^{(r)}$, $B_i^{(r)}$, $C_i^{(r)}$ correspond to $a_i^{(r)}$, $b_i^{(r)}$, $c_i^{(r)}$ on F_r and $L_i^{(r)}$ be the length of $A_i^{(r)}$ and put $L(r)=L_1^{(r)}+L_2^{(r)}+\cdots+L_{n_r}^{(r)}$. We will show that

$$\lim_{r\to\infty} A(r)=\infty \tag{3}$$

and there exists a sequence $\{r_n\}$ tending to infinity, such that

$$\frac{L(r)}{S(r)}\to 0\,, \qquad \text{when} \quad r=r_n\to\infty\,. \tag{4}$$

In the following we consider only such $r=r_n$.

We will prove (3) and (4) by modifying slightly Mr. Noshiro's[1)]

1) K. Noshiro: On the singularities of analytic functions. Japanese Journal of Mathematics, **17** (1940), 37–96.

proof. As well known, there exists a curve Γ in Δ tending to infinity, such that $\lim f(z)=0$, when z tends to infinity on Γ. Since one of $a_i^{(r)}$, $a_{i_0}^{(r)}$ say, intersects Γ, we have $\frac{\rho}{2} \leqq L_{i_0}^{(r)} \leqq L(r)$ for $r \geqq r_0$. Putting $\theta_r = a_1^{(r)} + a_2^{(r)} + \cdots + a_{n_r}^{(r)}$, we have for $r \geqq r_0$

$$\left(\frac{\rho}{2}\right)^2 \leqq \left(L(r)\right)^2 = \left(\int_{\theta_r} |f'(z)|\, r d\theta\right)^2 \leqq 2\pi r \int_{\theta_r} |f'(z)|^2 r d\theta = 2\pi r \left(\frac{dA(r)}{dr}\right)_-$$

where $\left(\frac{dA(r)}{dr}\right)_-$ denotes the derivative on the left. Hence if $a_i \geqq a_{i_0} \geqq r_0$

$$\frac{\rho^2}{8\pi}\int_{a_i}^{a_{i+1}} \frac{dr}{r} \leqq A(a_{i+1}) - A(a_i+0) \leqq A(a_{i+1}) - A(a_i)\,,$$

so that for $r \geqq a_{i_0}$, we have

$$\frac{\rho^2}{8\pi}\int_{a_{i_0}}^{r} \frac{dr}{r} \leqq A(r) - A(a_{i_0}) \leqq A(r)$$

whence $\lim_{r\to\infty} A(r) = \infty$. To prove (4) suppose that $L(r) > [A(r)]^{\frac{1}{2}+\frac{\varepsilon}{2}}$ $(\varepsilon > 0)$ on a set E. We take off all $\{a_i\}$ from E and the remaining set be denoted by E_0 which consists of open intervals $I_n = (a_n, a_n')$. Since on I_n, $\frac{A^{1+\varepsilon}(r)}{2\pi r} \leqq \frac{L(r)^2}{2\pi r} \leqq \frac{dA(r)}{dr}$, $\frac{1}{2\pi}\sum_n \int_{I_n} \frac{dr}{r} \leqq \sum_n \int_{I_n} \frac{dA}{A^{1+\varepsilon}} = \frac{1}{\varepsilon}\sum_n \left(\frac{1}{A^\varepsilon(a_n+0)} - \frac{1}{A^\varepsilon(a_n')}\right) \leqq \frac{1}{\varepsilon}\sum_n \left(\frac{1}{A^\varepsilon(a_n)} - \frac{1}{A^\varepsilon(a_n)}\right) \leqq \frac{1}{\varepsilon A^\varepsilon(a_1)}$ Hence $\int_E \frac{dr}{r} = \int_{E_0} \frac{dr}{r} < \infty$, so that there exists a sequence $r_n \to \infty$ such that $L(r_n) \leqq [A(r_n)]^{\frac{1}{2}+\frac{\varepsilon}{2}}$ or $\frac{L(r_n)}{S(r_n)} \to 0$, which proves (4). Let D_1, $D_2, \ldots, D_q$ $(q \geqq 2)$ be q simply connected domains in $|w| < \rho$ which have no common points with each other and D be any one of D_i.

The part of F_r which lies above D consists of a certain number of connected pieces which are of two types. Those pieces which have no relative boundaries in D are called islands, and the other pieces are called peninsulas. If all islands above D which are simply connected have at least μ sheets, then F_r is called μ-ply ramified above D. Then we will prove the following theorem.

Theorem. If F_r ramifies μ_k-ply above D_k $(k=1, 2, \ldots, q)$, then

$$\sum_{k=1}^{q}\left(1-\frac{1}{\mu_k}\right) \leqq 1 + \varliminf_{r\to\infty} \frac{A(r)}{S(r)} \leqq 2\,. \tag{5}$$

(5) contains K. Kunugui's theorem[2], that $\sum_{k=1}^{q}\left(1-\frac{1}{\mu_k}\right) \leqq 2$ and K.

2) K. Kunugui: Sur des fonctions méromorphes et uniformes, (which will appear in the Japanese Journal of Mathematics, **18** (1941)).

Noshiro's theorem[3], that $\sum_{k=1}^{q}\left(1-\frac{1}{\mu_k}\right)\leqq 1$, if $\varDelta$ is simply connected.

Proof. We apply Ahlfors' theory of covering surfaces on F_r. We denote the circular disc $|w|\leqq\rho$ by F_0 and Euler's characteristic of a domain D by $\rho(D)$ and $-\rho(D)=p(D)$ is called the simple multiplicity of D, which is $+1$, if D is simply connected, otherwise $p(D)\leqq 0$. We denote the sum of the simple multiplicities of all islands D^i of F_r above D_k by $\bar{p}(D_k)$.

First we take off from F_r all the peninsulas above $D_1+D_2+\cdots+D_q$, then there remains a certain number of connected pieces $\sum F'_r$. From $\sum F'_r$ we take off all the islands D^i above $D_1+D_2+\cdots+D_q$, there remains a certain number of connected pieces $\sum \bar{F}_r$, so that

$$\sum F'_r=\sum D^i+\sum \bar{F}_r\,.$$

Hence
$$\sum \rho(F'_r)=\sum \rho(D^i)+\sum \rho(\bar{F}_r)$$
or
$$\sum_{k=1}^{q}\bar{p}(D_k)=\sum \rho(\bar{F}_r)-\sum \rho(F'_r)=\sum \rho^+(\bar{F}_r)-\sum \rho(F'_r)-N_1(\bar{F}_r)$$
$$\geqq \sum \rho^+(\bar{F}_r)-\sum \rho^+(F'_r)\,,$$

where $N_1(\bar{F}_r)$ is the number of simply connected pieces in $\sum \bar{F}_r$. We put $\bar{F}_0=F_0-(D_1+D_2+\cdots+D_q)$. By Ahlfors' fundamental theorem[4]

$$\sum \rho^+(\bar{F}_r)\geqq (q-1)\,S(\bar{F}_0)-hL(r)$$

where h is a constant and $S(\bar{F}_0)=\dfrac{\text{area of }\sum \bar{F}_r}{\text{area of }\bar{F}_0}$ and by Ahlfors' first covering theorem, $S(\bar{F}_0)\geqq S(r)-hL(r)$, so that

$$\sum \rho^+(\bar{F}_r)\geqq (q-1)\,S(r)-hL(r)\,.$$

Considering the images of the peninsulas on the z-plane, we see easily, $\sum \rho^+(F'_r)\leqq \Lambda(r)$. Hence

$$\sum_{k=1}^{q}\bar{p}(D_k)\geqq (q-1)\,S(r)-\Lambda(r)-hL(r)\,.$$

If F ramifies μ_k-ply above D_k, then

$$\sum_{k=1}^{q}\frac{1}{\mu_k}\;S(D_k)\geqq \sum_{k=1}^{q}\bar{p}(D_k)\geqq (q-1)\,S(r)-\Lambda(r)-hL(r)\,.$$

Since by Ahlfors' first covering theorem, $S(D_k)\leqq S(r)+hL(r)$ and $L(r)=0\big(S(r)\big)$, we have

$$\sum_{k=1}^{q}\left(1-\frac{1}{\mu_k}\right)\leqq 1+\varliminf_{r\to\infty}\frac{\Lambda(r)}{S(r)}\,. \tag{6}$$

We will next prove $\varlimsup_{r\to\infty}\dfrac{\Lambda(r)}{S(r)}\leqq 1$.

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3) K. Noshiro l. c. p. 95.

4) L. Ahlfors: Zur Theorie der Überlagerungsflächen. Acta Math. **65** (1935).

Let F_r consist of a certain number of sheets $\Omega_1, \Omega_2, \dots, \Omega_N$ and Ω_i be such a sheet whose boundary consists of $C_i^{(r)}$ and one part of $A_i^{(r)}$ which form holes in Ω_i.

The sum of the lengths of the boundaries of holes be denoted by L_i', then $|\text{area of } \Omega_i - \pi\rho^2| < hL_i'$, where h is a constant depending on F_0 only. Hence summing up only for such Ω_i we have

$$S(r) \geqq \frac{\sum \text{area of } \Omega_i}{\pi\rho^2} \geqq \Lambda(r) - hL(r)\,, \tag{7}$$

so that

$$\overline{\lim_{r\to\infty}} \frac{\Lambda(r)}{S(r)} \leqq 1\,.$$

Remark. Similarly as Ahlfors, we can prove the defect relation

$$\sum_{k=1}^{q} \delta(D_k) + \sum_{k=1}^{q} \vartheta(D_k) \leqq 1 + \underline{\lim_{r\to\infty}} \frac{\Lambda(r)}{S(r)} \leqq 2\,.$$