# 89. *Note on the Kronecker Product of Representations of a Group.*

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(Comm. by T. TAKAGI, M.I.A., Nov. 12, 1941.)

The principal aim of this note is to prove the following Theorem 1. As an application we can prove a conjecture of R. Brauer-C. Nesbitt[1] (Th. 5).

Let $\mathfrak{G}$ be a group of finite order. We consider representations of $\mathfrak{G}$ in an arbitrary field $K$.

*Theorem 1. Let $\mathfrak{G}$ be a group of finite order and $R$ be its regular representation. If $V$ is a representation of $\mathfrak{G}$ of degree $m$, then*

$$V\times R\cong\begin{pmatrix} R & & & 0\\ & R & & \\ & & \ddots & \\ 0 & & & R\end{pmatrix}$$

*where $R$ appears $m$ times.*

Proof[2]. We denote by $G_1, G_2, \dots, G_t$ the elements of $\mathfrak{G}$. Let $G$ be an element of $\mathfrak{G}$. If $GG_i=G_j$, then

$$R(G)=\begin{array}{c} \\ \\ \\ j \\ \\ \\ \end{array}\!\!\begin{pmatrix} & \overset{i}{0} & \\ * & \vdots & * \\ 0\cdots & 1 & \cdot\cdot 0 \\ * & \cdot & * \\ & 0 & \end{pmatrix}$$

and

$$V(G)\times R(G)=\begin{pmatrix} & 0 & \\ * & \vdots & * \\ 0\cdot\cdot & V(G) & \cdot\ 0 \\ * & \cdot & * \\ & 0 & \end{pmatrix}.$$

If we put

$$P=\begin{pmatrix} V(G_1) & & & 0\\ & V(G_2) & & \\ & & \ddots & \\ 0 & & & V(G_t)\end{pmatrix}$$

then it follows from $GG_i=G_j$ that

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1) R. Brauer-C. Nesbitt, On the modular characters of groups, Ann. of Math. **42** (1941), p. 579.

2) If $R$ is completely reducible, we can easily see the validity of this theorem by comparing the characters of the representations.

$$P^{-1}\big(V(G)\times R(G)\big)P=\begin{pmatrix} & & 0 & & \\ & * & \vdots & * & \\ 0\cdots & & V(G_j^{-1}GG_i) & & \cdots 0 \\ & * & \vdots & * & \\ & & 0 & & \end{pmatrix}=\begin{pmatrix} & & 0 & & \\ & * & \vdots & * & \\ 0\cdots & & E_m & & \cdots 0 \\ & * & \vdots & * & \\ & & 0 & & \end{pmatrix}$$

where $E_m$ is the unit matrix of degree $m$. Hence we have

$$V\times R\cong E_m\times R\cong\begin{pmatrix} R & & 0 \\ & R & \\ & & \ddots \\ 0 & & R \end{pmatrix}$$

where $R$ appears $m$ times.

*Corollary. If $V$ and $W$ are representations of $\mathfrak{G}$ of the same degree, then $V\times R\cong W\times R$.*

*Theorem 2*[1]. *If $V$ is a representation of $\mathfrak{G}$ of degree $m$, then*

$$mR^{2)}\cong\begin{pmatrix} V & 0 \\ * & * \end{pmatrix}.$$

Proof. Since

$$R\cong\begin{pmatrix} 1 & 0 \\ * & * \end{pmatrix}$$

we obtain

$$V\times R\cong\begin{pmatrix} V & 0 \\ * & * \end{pmatrix}.$$

Our theorem now follows readily from Theorem 1.

Let $\mathfrak{G}$ be a group and $\mathfrak{H}$ be its subgroup. If $G\to V(G)$ is a representation of $\mathfrak{G}$, then $H\to V(H)$; $H\in\mathfrak{H}$ is a representation of $\mathfrak{H}$, which we denote by $V(\mathfrak{H})$. Now we can extend Theorem 1 to the following

*Theorem 3. Let $\mathfrak{G}$ be a group and $\mathfrak{H}$ be its subgroup of finite index. Let furthermore $R^*$ be the representation of $\mathfrak{G}$ induced from the 1-representation of $\mathfrak{H}$. If $V$ is a representation of $\mathfrak{G}$, then $V\times R^*\cong V^*(\mathfrak{H})$ where $V^*(\mathfrak{H})$ is the representation of $\mathfrak{G}$ induced from $V(\mathfrak{H})$.*

Let $F_1, F_2, \cdots, F_l$ be distinct absolutely irreducible representations of $\mathfrak{G}$ and $U_1, U_2, \cdots, U_l$ be corresponding directly indecomposable parts of $R$[3].

Since

$$V\times R\cong\begin{pmatrix} V\times U_1 & & 0 \\ & \ddots & \\ 0 & & V\times U_l \end{pmatrix}$$

we have from Theorem 1

1) K. Shoda has already proved this theorem. See K. Shoda, Über die Invarianten endlicher Gruppen linearer Substitutionen im Körper der Charakteristik $p$, Jap. J. of Math. **17** (1940).

2) We denote by $mR$ the representation of $\mathfrak{G}$ such that $R$ appears $m$ times on the diagonal.

3) If $R$ is completely reducible, then $U_\varkappa=F_\varkappa$.

*Theorem 4. Let V be a representation of* $\mathfrak{G}$. $V\times U_\varkappa$ *splits completely into* $U_1, U_2, \cdots, U_l$.

*Corollary. Let V and W be representations of* $\mathfrak{G}$ *which have the same irreducible constituents. Then* $V\times U_\varkappa \cong W\times U_\varkappa$.

Proof. The characters of $U_\lambda$ $(\lambda=1, 2, \cdots, l)$ are linearly independent. Hence the corollary is immediate.

Denote the character of $F_\varkappa$ and $U_\varkappa$ by $\varphi^{(\varkappa)}$ and $\eta^{(\varkappa)}$ respectively. From $\varphi^{(\varkappa)}\cdot\varphi^{(\lambda)}=\sum_\mu a_{\varkappa\lambda\mu}\varphi^{(\mu)}$ it follows that[1] $\eta^{(\mu)}\cdot\varphi^{(\lambda')}=\sum_\varkappa a_{\varkappa\lambda\mu}\eta^{(\varkappa)}$ where $\varphi^{(\lambda')}$ is the character of the representation $F_{\lambda'}$ contragredient to $F_\lambda$. We obtain from Theorem 4

*Theorem 5. Let* $a_{\varkappa\lambda\mu}$ *be the multiplicity of* $F_\mu$ *as irreducible constituent of* $F_\varkappa\times F_\lambda$. $U_\mu\times F_{\lambda'}$ *splits completely into* $U_1, U_2, \cdots, U_l$ *where* $U_\varkappa$ *appears* $a_{\varkappa\lambda\mu}$ *times.*

*Corollary. Let* $c_{\mu\nu}$ *be the multiplity of* $F_\nu$ *as irreducible constituent of* $U_\mu$. $U_1$ *appears* $c_{\mu\nu}$ *times in* $U_\mu\times U_{\nu'}$ *where* $U_{\nu'}$ *is the representation contragredient to* $U_\nu$.

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1) See Brauer-Nesbitt, l. c.