97. *On the Generalized Loxodromes in the Conformally Connected Manifold.*

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In a previous paper[1], we have defined a conformal arc length σ on the curves in the conformally connected manifold and have proved the Frenet fomulae

$$
(1) \qquad \left\{
\begin{aligned}
\frac{d}{d\sigma}\underset{(0)}{A} &= \underset{(1)}{A}, \\
\frac{d}{d\sigma}\underset{(1)}{A} &= \lambda\underset{(0)}{A} + \underset{(2)}{A}, \\
\frac{d}{d\sigma}\underset{(2)}{A} &= \lambda\underset{(1)}{A} + \underset{(3)}{A}, \\
\frac{d}{d\sigma}\underset{(3)}{A} &= \underset{(0)}{A} + \overset{4}{\lambda}\underset{(4)}{A}, \\
\frac{d}{d\sigma}\underset{(4)}{A} &= -\overset{4}{\lambda}\underset{(3)}{A} + \overset{5}{\lambda}\underset{(5)}{A}, \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
\frac{d}{d\sigma}\underset{(\infty)}{A} &= -\overset{\infty}{\lambda}\underset{(n)}{A}.
\end{aligned}
\right.
$$

The conformal arc length does not exist for the generalized circles.

The most simple curves having the conformal arc length are the ones for which we have

$$
(2) \qquad \lambda = \overset{4}{\lambda} = \overset{5}{\lambda} = \cdots = \overset{n}{\lambda} = \overset{\infty}{\lambda} = 0 .
$$

We shall, first, consider the properties of these curves.

§ 1. Substituting the equations (2) in the Frenet formulae (1), we have

$$
(3) \qquad \frac{d}{d\sigma}\underset{(0)}{A} = \underset{(1)}{A}, \quad \frac{d}{d\sigma}\underset{(1)}{A} = \underset{(2)}{A}, \quad \frac{d}{d\sigma}\underset{(2)}{A} = \underset{(3)}{A}, \quad \frac{d}{d\sigma}\underset{(3)}{A} = \underset{(0)}{A},
$$

which shows that, if we develop the curve on the tangent conformal space at $\underset{(0)}{A}$, we shall obtain a curve lying always on a two-dimensional sphere determined by $\underset{(0)}{A}, \underset{(1)}{A}, \underset{(2)}{A}$ and $\underset{(3)}{A}$.

Considering always the development of the curve on the tangent conformal space at a fixed point of the conformally connected manifold, we can treat such curves as if they were on a two-dimensional flat conformal space.

1) K. Yano and Y. Mutô: On the conformal arc length, Proc. **17** (1941), 318-322.

Now, putting

$$
\left\{\begin{aligned} P &= \underset{(0)}{A}+\underset{(1)}{A}+\underset{(2)}{A}+\underset{(3)}{A}, \\ Q &= \underset{(0)}{A}-\underset{(1)}{A}+\underset{(2)}{A}-\underset{(3)}{A}, \end{aligned}\right. \tag{4}
$$

we have

$$
\frac{d}{d\sigma}P=P \quad\text{and}\quad \frac{d}{d\sigma}Q=-Q, \tag{5}
$$

consequently, we can see that P and Q are two fixed points.

The circle passing through the three points $\underset{(0)}{A}$, P and Q being $\underset{(1)}{A}-\underset{(3)}{A}$, the angle θ between the curve and $\underset{(1)}{A}-\underset{(3)}{A}$ is given by

$$
\sin\theta=\frac{\underset{(1)}{A}(\underset{(1)}{A}-\underset{(3)}{A})}{\sqrt{\underset{(1)}{A}\underset{(1)}{A}}\sqrt{(\underset{(1)}{A}-\underset{(3)}{A})(\underset{(1)}{A}-\underset{(3)}{A})}}=\frac{1}{\sqrt{2}}
$$

from which we find $\theta=\dfrac{\pi}{4}$. Thus we have the

Theorem I. The curve whose all conformal curvatures $\lambda, \overset{4}{\lambda}, \dots, \overset{n}{\lambda}, \overset{\infty}{\lambda}$ *vanish cuts the circles passing through two fixed points always by the fixed angle* $\dfrac{\pi}{4}$, *consequently, it is a loxodrome.*

The four points $\underset{(0)}{A}$, $\underset{(2)}{A}$, P and Q lying on the circle $\underset{(1)}{A}-\underset{(3)}{A}$, the points $\underset{(0)}{A}$ and $\underset{(2)}{A}$ are harmonically separated by the points P and Q on this circle.

We shall now find the differential equations of these generalized loxodromes.

Differentiating the equation

$$
\underset{(0)}{A}=\left(\frac{d\sigma}{ds}\right)A_0, \tag{6}
$$

and making use of the formulae

$$
\left\{\begin{aligned} dA_0 &= du^i A_i, \\ dA_j &= \Pi^0_{jk}du^k A_0+\Pi^i_{jk}du^k A_i+\Pi^\infty_{jk}du^k A_\infty, \\ dA_\infty &= \Pi^i_{\infty k}du^k A_i, \end{aligned}\right.
$$

which define the conformal connection, we find successively

$$
\underset{(1)}{A}=\frac{\dfrac{d^2\sigma}{ds^2}}{\dfrac{d\sigma}{ds}}A_0+\frac{du^i}{ds}A_i, \tag{7}
$$

$$
(8)\qquad \underset{(2)}{A}=\left[\frac{\dfrac{d^3\sigma}{ds^3}}{\left(\dfrac{d\sigma}{ds}\right)^2}-\frac{\left(\dfrac{d^2\sigma}{ds^2}\right)^2}{\left(\dfrac{d\sigma}{ds}\right)^3}+\frac{\Pi^0_{jk}\dfrac{du^j}{ds}\dfrac{du^k}{ds}}{\dfrac{d\sigma}{ds}}\right]A_0
+\left[\frac{\dfrac{\delta^2u^i}{\delta s^2}}{\dfrac{d\sigma}{ds}}+\frac{\dfrac{d^2\sigma}{ds^2}\dfrac{du^i}{ds}}{\left(\dfrac{d\sigma}{ds}\right)^2}\right]A_i+\frac{1}{\dfrac{d\sigma}{ds}}A_\infty,
$$

$$
(9)\qquad \underset{(3)}{A}=\frac{1}{\left(\dfrac{d\sigma}{ds}\right)^2}\left[g_{jk}\frac{\delta^3u^j}{\delta s^3}\frac{\delta^2u^k}{\delta s^2}+\Pi^0_{jk}\frac{\delta^2u^j}{\delta s^2}\frac{du^k}{ds}\right]A_0
+\frac{1}{\left(\dfrac{d\sigma}{ds}\right)^2}\left[\frac{\delta^3u^i}{\delta s^3}+\frac{du^i}{ds}g_{jk}\frac{\delta^2u^j}{\delta s^2}\frac{\delta^2u^k}{\delta s^2}
-\frac{du^i}{ds}\Pi^0_{jk}\frac{du^j}{ds}\frac{du^k}{ds}+\Pi^i_{\infty k}\frac{du^k}{ds}\right]A_i{}^{*)},
$$

where we have denoted by $\delta/\delta s$ the covariant differentiation along the curve with respect to the Christoffel symbols $\Pi^i_{jk}=\{{}^{\,i}_{jk}\}$.

Putting

$$
(10)\qquad V^i=\frac{\delta^3u^i}{\delta s^3}+\frac{du^i}{ds}g_{jk}\frac{\delta^2u^j}{\delta s^2}\frac{\delta^2u^k}{\delta s^2}-\frac{du^i}{ds}\Pi^0_{jk}\frac{du^j}{ds}\frac{du^k}{ds}+\Pi^i_{\infty k}\frac{du^k}{ds},
$$

we have, from (9),

$$
(11)\qquad \underset{(3)}{A}=\frac{1}{\left(\dfrac{d\sigma}{ds}\right)^2}\left[g_{jk}V^j\frac{\delta^2u^k}{\delta s^2}A_0+V^iA_i\right].
$$

Differentiating once more the equation (11) along the curve, we have, in consequence of (3),

$$
(12)\qquad \frac{d}{d\sigma}\underset{(3)}{A}=\underset{(0)}{A}=\left[-\frac{2\dfrac{d^2\sigma}{ds^2}}{\left(\dfrac{d\sigma}{ds}\right)^4}g_{jk}V^j\frac{\delta^2u^k}{\delta s^2}
+\frac{1}{\left(\dfrac{d\sigma}{ds}\right)^3}\left(g_{jk}\frac{\delta V^j}{\delta s}\frac{\delta^2u^k}{\delta s^2}+g_{jk}V^j\frac{\delta^3u^k}{\delta s^3}+\Pi^0_{jk}V^j\frac{du^k}{ds}\right)\right]A_0
+\left[-\frac{2\dfrac{d^2\sigma}{ds^2}}{\left(\dfrac{d\sigma}{ds}\right)^4}V^i+\frac{1}{\left(\dfrac{d\sigma}{ds}\right)^3}\left(g_{jk}V^j\frac{\delta^2u^k}{\delta s^2}\frac{du^i}{ds}+\frac{\delta V^i}{\delta s}\right)\right]A_i,
$$

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*) We have used the relations $0=\lambda=-\{t,\sigma\}=-[\{t,s\}-\{\sigma,s\}]\left(\dfrac{ds}{d\sigma}\right)^2$ and $\{\sigma,s\}=\{t,s\}=\dfrac{1}{2}g_{jk}\dfrac{\delta^2u^j}{\delta s^2}\dfrac{\delta^2u^k}{\delta s^2}-\Pi^0_{jk}\dfrac{du^j}{ds}\dfrac{du^k}{ds}$ for the reduction.

from which we obtain

$$(13)\quad g_{jk}\frac{\delta V^j}{\delta s}\frac{\delta^2 u^k}{\delta s^2}+g_{jk}V^j\frac{\delta^3 u^k}{\delta s^3}+\Pi^0_{jk}V^j\frac{du^k}{ds}-\frac{2\dfrac{d^2\sigma}{ds^2}}{\dfrac{d\sigma}{ds}}g_{jk}V^j\frac{\delta^2 u^k}{\delta s^2}=\left(\frac{d\sigma}{ds}\right)^4$$

and

$$(14)\quad \frac{\delta V^i}{\delta s}+g_{jk}V^j\frac{\delta^2 u^k}{\delta s^2}\frac{du^i}{ds}=\frac{2\dfrac{d^2\sigma}{ds^2}}{\dfrac{d\sigma}{ds}}V^i.$$

But, as we can see from (10), we have

$$g_{jk}V^j\frac{du^k}{ds}=0$$

which gives, on covariant differentiation,

$$(15)\quad g_{jk}V^j\frac{\delta^2 u^k}{\delta s^2}=-g_{jk}\frac{\delta V^j}{\delta s}\frac{du^k}{ds}.$$

Substituting these equations in (14), we find

$$(16)\quad \frac{\delta V^i}{\delta s}-\frac{du^i}{ds}g_{jk}\frac{\delta V^j}{\delta s}\frac{du^k}{ds}=\frac{2\dfrac{d^2\sigma}{ds^2}}{\dfrac{d\sigma}{ds}}V^i$$

or

$$(17)\quad \frac{Dv^i}{Ds}=0$$

where D/Ds is a conformal covariant derivative defined in a previous paper[1], and

$$(18)\quad v^i=\frac{V^i}{\left(\dfrac{d\sigma}{ds}\right)^2}$$

σ being defined by

$$(19)\quad \left(\frac{d\sigma}{ds}\right)^4=g_{jk}V^jV^k.$$

If the equations (16) hold good, the left-hand member of (13) will be written as

$$\frac{2\dfrac{d^2\sigma}{ds^2}}{\dfrac{d\sigma}{ds}}g_{jk}V^j\frac{\delta^2 u^k}{\delta s^2}+g_{jk}V^jV^k-\frac{2\dfrac{d^2\sigma}{ds^2}}{\dfrac{d\sigma}{ds}}g_{jk}V^j\frac{\delta^2 u^k}{\delta s^2}$$

then, in consequence of (19), the equations (13) are automatically satisfied.

1) K. Yano: Sur la connexion de Weyl-Hlavatý et ses applications à la géométrie conforme, Proc. Physico-Math. Soc. Japan. Vol. **22** (1940), 595–621.

So that, (17) are the differential equations of the generalized loxodromes to be obtained.

§ 2. Next, we shall consider the curves for which we have

$$(20) \qquad \lambda = \text{const.} \quad \overset{4}{\lambda} = \overset{5}{\lambda} = \cdots = \overset{n}{\lambda} = \overset{\infty}{\lambda} = 0\,.$$

For these curves, we have, from (1),

$$(21) \qquad \frac{d}{d\sigma}\underset{(0)}{A} = \underset{(1)}{A}\,, \quad \frac{d}{d\sigma}\underset{(1)}{A} = \lambda\underset{(0)}{A} + \underset{(2)}{A}\,, \quad \frac{d}{d\sigma}\underset{(2)}{A} = \lambda\underset{(1)}{A} + \underset{(3)}{A}\,, \quad \frac{d}{d\sigma}\underset{(3)}{A} = \underset{(0)}{A}\,,$$

consequently, we can, as in § 1, treat the curves as if they were in a two-dimensional conformal space. In order that the circle

$$(22) \qquad \alpha_0\underset{(0)}{A} + \alpha_1\underset{(1)}{A} + \alpha_2\underset{(2)}{A} + \alpha_3\underset{(3)}{A}$$

where α_0, α_1, α_2 and α_3 are constants, be a fixed circle, the constants α_0, α_1, α_2 and α_3 must satisfy the equations

$$(23) \qquad \alpha_1\lambda + \alpha_3 = \rho\alpha_0\,, \quad \alpha_2\lambda + \alpha_0 = \rho\alpha_1\,, \quad \alpha_1 = \rho\alpha_2\,, \quad \alpha_2 = \rho\alpha_3$$

where ρ is a proportional constant.

Eliminating α_0, α_1, α_2 and α_3 from these equations, we have

$$(24) \qquad \rho^4 - 2\lambda\rho^2 - 1 = 0$$

which gives two real values of ρ

$$(25) \qquad \begin{cases} \rho = +\sqrt{\lambda + \sqrt{\lambda^2+1}}\,, \\ \rho' = -\sqrt{\lambda + \sqrt{\lambda^2+1}}\,. \end{cases}$$

Corresponding to these values of ρ, we have

$$\alpha_0 : \alpha_1 : \alpha_2 : \alpha_3 = \sqrt{\lambda^2+1}\cdot\rho : \rho^2 : \rho : 1\,,$$

and

$$\alpha_0 : \alpha_1 : \alpha_2 : \alpha_3 = \sqrt{\lambda^2+1}\cdot\rho' : \rho'^2 : \rho' : 1 = -\sqrt{\lambda^2+1}\cdot\rho : \rho^2 : -\rho : 1\,.$$

Consequently, the two circles defined by

$$(26) \qquad \begin{cases} P = +\sqrt{\lambda^2+1}\,\rho\underset{(0)}{A} + \rho^2\underset{(1)}{A} + \rho\underset{(2)}{A} + \underset{(3)}{A}\,, \\ Q = -\sqrt{\lambda^2+1}\,\rho\underset{(0)}{A} + \rho^2\underset{(1)}{A} - \rho\underset{(2)}{A} + \underset{(3)}{A}\,, \end{cases}$$

are both fixed circles. Moreover, we can easily verify that

$$(27) \qquad P\cdot P = 0 \quad \text{and} \quad Q\cdot Q = 0\,,$$

that is to say that the P and Q are both point-circles.

The circle passing through the three points $\underset{(0)}{A}$, P and Q being $\underset{(1)}{A} - \rho^2\underset{(3)}{A}$, the angle θ $(0 \leqq \theta \leqq \pi)$ at $\underset{(0)}{A}$ between the curve and this circle is given by

$$\text{(28)} \qquad \sin\theta = \frac{\underset{(1)}{A}(\underset{(1)}{A} - \rho^2 \underset{(3)}{A})}{\sqrt{\underset{(1)}{A}\underset{(1)}{A}}\sqrt{(\underset{(1)}{A} - \rho^2 \underset{(3)}{A})(\underset{(1)}{A} - \rho^2 \underset{(3)}{A})}} = \frac{1}{\sqrt{\rho^4+1}}$$

$$= \frac{1}{\sqrt{2(\lambda^2+1)+2\lambda\sqrt{\lambda^2+1}}}\,.$$

Putting

$$\text{(29)} \qquad \lambda = \tan\varphi \qquad \left(-\frac{\pi}{2} \leqq \varphi \leqq \frac{\pi}{2}\right)$$

we have, from (28),

$$\sin\theta = \frac{1}{\sqrt{2\sec^2\varphi + 2\tan\varphi\sec\varphi}} = \frac{1}{\sqrt{2}}\left(\cos\frac{\varphi}{2} - \sin\frac{\varphi}{2}\right) = \sin\left(\frac{\pi}{4} - \frac{\varphi}{2}\right)$$

hence,

$$\text{(29)} \qquad \theta = \frac{\pi}{4} - \frac{\varphi}{2}\,.$$

Then, we have the

Theorem II. The curve whose conformal curvature $\lambda = $ *const. and* $\overset{4}{\lambda} = \overset{5}{\lambda} = \cdots = \overset{\infty}{\lambda} = 0$, *cuts all the circles passing through two fixed points by the fixed angle* $\frac{\pi}{4} - \frac{\varphi}{2}$ *where* $\tan\varphi = \lambda$, *consequently it is a loxodrome.*

The four points $\underset{(0)}{A}$, $\underset{(2)}{A}$, P and Q lying on the circle $\underset{(1)}{A} - \rho^2 \underset{(3)}{A}$, the points $\underset{(0)}{A}$ and $\underset{(2)}{A}$ are harmonically separated by the points P and Q on this circle.