PAPERS COMMUNICATED

1. Note on Lattice-ordered Groups.

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(Comm. by T. TAKAGI, M.I.A., Jan. 12, 1942.)

By a lattice-ordered group, or briefly a lattice-group, we mean a (not necessarily abelian) group which is at the same time a lattice such that the order relation is preserved under left and right multiplication; $a \leqq b$ implies $ac \leqq bc$ and $ca \leqq cb$; we have $ac \curlyvee\!\!\!\!\curlywedge bc = (a \curlyvee\!\!\!\!\curlywedge b)c$, $ca \curlyvee\!\!\!\!\curlywedge cb = c(a \curlyvee\!\!\!\!\curlywedge b)$ too.

Abelian lattice-groups[1], particularly so-called vector lattices[2], have been studied by many authors. The present short note[3] is to give some simple remarks concerning mainly with non-abelian lattice-groups. We shall begin with elementary observations about homomorphisms. We shall then show that Lorenzen's main theorem for abelian lattice-groups can be transfered to the non-abelian case with minor modifications. However, this does not give, contrary to Clifford's abelian case, a representation of the lattice-group by linearly ordered ones: it gives merely a representation of the lattice-group by linearly ordered systems of cosets with respect to its subgroups. It follows readily that every lattice-group is, considered as a lattice, distributive. This fact, however, can easily be seen also by modifying somewhat the well-known proofs to the distributivity of abelian lattice-groups[4]. The structure of lattice-groups satisfying the conditional (=weak) maximum condition is very simple and rather trivial; they are necessarily abelian[5][6]. We shall also observe that a recent result by Yosida-Fukamiya[7] concerning

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1) R. Dedekind, Über Zerlegung von Zahlen durch ihre grössten gemeinschaftlichen Teiler (Ges. Werke, Bd. 2, XXVIII); P. Lorenzen, Abstrakte Begründung der multiplikativen Idealtheorie, Math. Zeitschr. **45** (1939); A. H. Clifford, Partially ordered abelian groups, Ann. Math. **41** (1940).

2) L. V. Kantorovitch, Lineare halbgeordnete Räume, Mat. Shornik **2** (1937); H. Freudenthal, Teilweise geordnete Moduln, Proc. Amsterdam **39** (1936). For some of more recent literatures see the references in G. Birkhoff, Lattice theory, New York (1940); K. Yosida, On vector lattice with a unit, Proc. **17** (1941).

3) I want to express my thanks to Mr. K. Yosida for the useful remarks he gave me during the preparation of the present note.

4) For abelian case see Dedekind, l. c., Freudenthal, l. c. and Birkhoff, l. c.

5) As a matter of fact, the essential feature of this fact is contained already in the commutativity of two-sided ideals in the arithmetical theory of algebras, non-commutative polynomicals and non-commutative semi-groups. Besides early works by E. Artin, O. Ore and others, cf. K. Asano, Arithmetische Idealtheorie in nichtkommutativen Ringen, Jap. Journ. Math. **16** (1939); Y. Kawada-K. Kondo, Idealtheorie in nicht-kommutativen Halbgruppen, ibid. **16** (1939); T. Nakayama, A note on the elementary divisor theory in non-commutative domains, Bull. Amer. Math. Soc. **44** (1938).

6) This is a very simple; and trivial, special case of G. Birkhoff's conjecture that conditionally complete lattice-groups will always be abelian. The conjecture was communicated to me by S. Kakutani.

7) K. Yosida-M. Fukamiya, On vector lattice with a unit, II., Proc. **17** (1941).

vector-lattices with Archimedean units may be obtained also by means of Lorenzen-Clifford's theorem.

§ **1.** Let G be a lattice-group. A subgroup H of G shall be called an *m-subgroup*, if $h_1, h_2 \in H$ and $h_1 \cap h_2 \leqq x \leqq h_1 \cup h_2$ imply $x \in H$. For an m-subgroup H we define the join $Ha \cup Hb$ of two right cosets Ha, Hb mod. H to be the coset $H(a \cup b)$ of $a \cup b$; this does not depend on the choice of the representatives, because if $h_1, h_2 \in H$ then $(h_1 \cap h_2)(a \cup b) \leqq h_1 a \cup h_2 b \leqq (h_1 \cup h_2)(a \cup b)$, $h_1 \cap h_2 \leqq (h_1 a \cup h_2 b)(a \cup b)^{-1} \leqq h_1 \cup h_2$ whence $(h_1 a \cup h_2 b)(a \cup b)^{-1} \in H$. Similarly we put $Ha \cap Hb = H(a \cap b)$. Then the totality $(G/H)_r$ of right cosets mod. H becomes a lattice, and $a \rightarrow Ha$ is a lattice-homomorphism of G onto $(G/H)_r$. It is evident that conversely if $a \rightarrow Ha$ is a lattice-homomorphism then H is an m-subgroup. If in particular H is an invariant m-subgroup then the factor group G/H is, under the compositions $\cup$, $\cap$ defind above, a lattice-group, and $a \rightarrow Ha$ is a lattice-group-homomorphism of G onto G/H[1]; and, conversely every lattice-group-homomorphism of G is given rise by a suitable invariant m-subgroup.

In order that the lattice $(G/H)_r$ be linearly ordered, it is necessary and sufficient that $a \leqq 1$, $b \leqq 1$, $a \notin H$, $b \notin H$ imply $a \cup b \notin H$; 1 being the unit element of G. The necessity is evident. But, if $(G/H)_r$ is not linearly ordered, there exist a_1 and b_1 such that $H(a_1 \cup b_1) > Ha_1$, Hb_1 and then for $a = a_1(a_1 \cup b_1)^{-1}$, $b = b_1(a_1 \cup b_1)^{-1}$ we have $a, b \leqq 1$, $a, b \notin H$ but $a \cup b = 1 \in H$.

§ **2.** An element a of G is called integral, when $a \leqq 1$. We denote the totality of integral elements by $\mathfrak{g}$. $a\mathfrak{g} = \mathfrak{g}a$ for every $a \in G$. By an *s-ideal*[2] we mean a subset $\mathfrak{a}$ of G bounded from above such that $\mathfrak{g}\mathfrak{a}\mathfrak{g} (= \mathfrak{g}\mathfrak{a} = \mathfrak{a}\mathfrak{g}) \leqq$ (whence $=$) $\mathfrak{a}$; this last condition is the M-closedness in the lattice-theory. An s-ideal $\mathfrak{a}$ is called a *t-ideal* when a, $b \in \mathfrak{a}$ implies $a \cup b \in \mathfrak{a}$; it is a lattice-ideal of G. An (s- or t-) ideal $\mathfrak{a}$ is called integral when $\mathfrak{a} \leqq \mathfrak{g}$. An integral ideal $\mathfrak{p}$ is said to be prime if $a_1, a_2 \notin \mathfrak{p}$ implies $a_1 a_2 \notin \mathfrak{p}$; it is evident that then $a_1 \cap a_2 \notin \mathfrak{p}$ too. If $\mathfrak{p}$ is a prime ideal and $a \in \mathfrak{g} - \mathfrak{p}$ then $a\mathfrak{p} = \mathfrak{p}a$, because $a\mathfrak{p}a^{-1}a \leqq \mathfrak{p}$ whence $a\mathfrak{p}a^{-1} \leqq \mathfrak{p}$ and similarly $a\mathfrak{p}a^{-1} \geqq \mathfrak{p}$. A maximal (integral) t-ideal $\mathfrak{p}$ is always prime; for, if $a_1, a_2 \in \mathfrak{g} - \mathfrak{p}$ then $p_i \cup a_i = 1$ for suitable $p_i \in \mathfrak{p}$ $(i = 1, 2)$ and $1 = (p_1 p_2 \cup p_1 a_2 \cup a_1 p_2) \cup a_1 a_2$ whence $a_1 a_2 \notin \mathfrak{p}$.

Let $\mathfrak{p}$ be a prime s-ideal, and denote the totality of the elements of the form ac^{-1} $(a \in \mathfrak{g},\ c \in \mathfrak{g} - \mathfrak{p})$ by $\mathfrak{g}_{\mathfrak{p}}$; observe that $ac^{-1} = c^{-1}a'$ where $a' = cac^{-1} \in \mathfrak{g}$. $\mathfrak{g}_{\mathfrak{p}}$ is a semi-group, since $a_1 c_1^{-1} a_2 c_2^{-1} = a_1(c_1^{-1} a_2 c_1)(c_2 c_1)^{-1}$ and here $a_1(c_1^{-1} a_2 c_1) \in \mathfrak{g}$, $c_2 c_1 \in \mathfrak{g} - \mathfrak{p}$ when $a_1, a_2 \in \mathfrak{g}$, $c_1, a_2 \in \mathfrak{g} - \mathfrak{p}$. Furthermore, it is a lattice-ideal of G. For, $d = c_1 \cap c_2 \notin \mathfrak{p}$ too and $a_1 c_1^{-1} \cup a_2 c_2^{-1} = \big(a_1({}_1^{-1}d) \cup a_2(c_2^{-1}d)\big)d^{-1}$. Thus the totality $\mathfrak{g}_{\mathfrak{p}}^{(-1)}$ of the inverses of the elements in $\mathfrak{g}_{\mathfrak{p}}$ is a semi-group which is, at the same time, a dual lattice-ideal of G. Hence the intersection $H_{\mathfrak{p}} = \mathfrak{g}_{\mathfrak{p}} \cap \mathfrak{g}_{\mathfrak{p}}^{(-1)}$, that is, the set of units of $\mathfrak{g}_{\mathfrak{p}}$, is an m-subgroup of G. $\mathfrak{g}_{\mathfrak{p}}$ is the M-closure of $H_{\mathfrak{p}}$, as one readily sees.

Now, if $\mathfrak{p}$ is a prime (not only s- but) t-ideal, then the lattice

1) For abelian case cf. G. Birkhoff's book, l. c. § 136.

2) See Lorenzen, l. c.

$(G/H_\mathfrak{p})_r$ of right cosets mod. $H_\mathfrak{p}$ is linearly ordered; this is equivalent to saying that for every x in G at least one of x and x^{-1} lies in $\mathfrak{g}_\mathfrak{p}$. To see this, observe that a non-unit f of $\mathfrak{q}_\mathfrak{p}$ (that is, an element f of $\mathfrak{g}_\mathfrak{p}$ whose inverse f^{-1} is not in $\mathfrak{g}_\mathfrak{p}$) is of a form $f=pc^{-1}$ ($p\in\mathfrak{p}$, $c\in\mathfrak{g}-\mathfrak{p}$), and that the join $p_1c_1^{-1}\cup p_2c_2^{-1}=\big(p_1(c_1^{-1}d)\cup p_2(c_2^{-1}d)\big)d^{-1}$ ($d=c_1\cap c_2$) of two non-units $p_1c_1^{-1}$ and $p_2c_2^{-1}$ is again a non-unit of $\mathfrak{g}_\mathfrak{p}$, since $d\in\mathfrak{g}-\mathfrak{p}$. The assertion follows thus immediately from a remark at the end of § 1.

Now, let $x\notin\mathfrak{g}$. Then $(1\cup x)^{-1}<1$. There exists a maximal t-ideal $\mathfrak{p}$ which contains $(1\cup x)^{-1}$; $\mathfrak{p}$ is prime. Then $x\notin\mathfrak{g}_\mathfrak{p}$, since otherwise $1\cup x\in\mathfrak{g}_\mathfrak{p}$.

From these considerations we have[1)]

Theorem 1. Let G be an arbitrary lattice-group. The semi-group $\mathfrak{g}$ of its integral elements is the intersection $\mathfrak{g}=\cap\,\mathfrak{g}_\mathfrak{p}$ of all the quotient-semi-group $\mathfrak{g}_\mathfrak{p}$ of $\mathfrak{g}$ with respect to maximal t-ideals $\mathfrak{p}$. $\mathfrak{g}_\mathfrak{p}$ has the property that $x\notin\mathfrak{g}_\mathfrak{p}$ implies $x^{-1}\in\mathfrak{g}_\mathfrak{p}$.

Theorem 1′. $a\rightarrow(\ldots\ldots, H_\mathfrak{p}a, \ldots\ldots)$ is a faithful lattice-homomorphic mapping of G into the direct product $\cdots\cdots\times\mathfrak{C}_\mathfrak{p}\times\cdots\cdots$ of the linearly ordered lattices $\mathfrak{C}_\mathfrak{p}=(G/H_\mathfrak{p})_r$ of cosets. The homomorphic mapping is preserved under right-hand side multiplication by group elements.

§ **3.** From Theorem 1′ follows immediately

Theorem 2. Every group-lattice is, considered as a lattice, distributive.

This fact can, however, easily be seen also by modifying a little the well-known proofs to the distributivity of abelian lattice-groups. For instance, it is sufficient to show that relative complementation is unique in G[2)]. Let, therefore, $a\cup x=a\cup y$, $a\cap x=a\cap y$ in G. Then $1\cup a^{-1}x=1\cup a^{-1}y$, $1\cap a^{-1}x=1\cap a^{-1}y$. On multiplying the equalities side by side, we get $a^{-1}x=(1\cup a^{-1}x)(1\cap a^{-1}x)=(1\cup a^{-1}y)(1\cup a^{-1}y)=a^{-1}y$. Hence $x=y$, and the distributivity of G is shown[3)].

§ **4.** Assume in this section that our lattice-group G satisfies the conditional maximum condition: every ascending chain bounded from above is finite. (Then the conditional minimum condition is fulfilled too). Every (s- or t-) ideal $\mathfrak{a}$ has a maximal element, and it is a unique maximal element if $\mathfrak{a}$ is a t-ideal. Thus every t-ideal is principal, $\mathfrak{a}=a\mathfrak{g}=\mathfrak{g}a$. Let in particular $\mathfrak{p}=p\mathfrak{g}=\mathfrak{g}p$ be a prime t-ideal. Evidently $p\mathfrak{p}=\mathfrak{p}p$. But, for every x in G there is an n such that $1\geqq xp^{-n}\not\leqq p$ and xp^{-n} is, as was observed before, commutative with $\mathfrak{p}$. Hence $x\mathfrak{p}=\mathfrak{p}x$ too, and the m-subgroup $H_\mathfrak{p}$ is invariant.

The linearly ordered group $\mathfrak{C}_\mathfrak{p}=G/H_\mathfrak{p}$ satisfies the conditional maximum condition, and therefore, it is an (infinite) cyclic group generated by its largest integral element. It is now easy to obtain

Theorem 3. A lattice-group satisfying the conditional maximum condition is isomorphic, as a lattice-group, with a restricted direct pro-

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1) Cf. Lorenzen, l. c., Satz 11 and Clifford, l. c., Theorem 2.

2) Cf. Birkhoff, l. c., § 137.

3) Similarly, Freudenthal's proof can easily be modified so as to apply to the non-abelian case. The same is the case also for H. Nakano's proof (Zenkoku-Sizyo-Sugaku-Danwakwai **228** (1941), in Japanese).

duct of a (finite or infinite) number of linearly ordered groups isomorphic to the (additive) group of rational integers. In particular, it is abelian.

§ **5.** Consider now a (not necessarily abelian) linearly ordered lattice-group $\mathfrak{C}$ with an Archimedean unit I; for every $x \in \mathfrak{C}$ there exists an n so that $I^n \leqq x \leqq I^{-n}$. After Yosida-Fukamiya, let us call an element x *nilpotent* if $I < x^i < I^{-1}$ for all $i = 1, 2, \ldots\ldots$. The totality $\mathfrak{N}$ of nilpotent elements is an m-subgroup of $\mathfrak{C}$, because if $x \leqq y$ then $x^{2i} \leqq (xy)^i \leqq y^{2i}$. On the other hand, any m-subgroup of $\mathfrak{C}$ not coinciding with $\mathfrak{C}$ consists only of nilpotent elements. Thus $\mathfrak{N}$ is the unique maximal m-subgroup of $\mathfrak{C}$.

Now, let G be an abelian lattice-group with an Archimedean unit I. Then, for each maximal t-ideal $\mathfrak{p}$, the coset $H_{\mathfrak{p}}I$ of I mod. $H_{\mathfrak{p}}$ is an Archimedean unit of the linearly ordered group $G/H_{\mathfrak{p}}$. It is further evident that in order that an element x of G be nilpotent (that is, $I < x^i < I^{-1}$ for $i = 1, 2, \ldots\ldots$) it is necessary and sufficient that $H_{\mathfrak{p}}x$ is nilpotent in $\mathfrak{C}_{\mathfrak{p}} = G/H_{\mathfrak{p}}$ for every $\mathfrak{p}$. Therefore, if x is not nilpotent, then for a suitable maximal t-ideal $\mathfrak{p}$ $H_{\mathfrak{p}}x \notin \mathfrak{N}_{\mathfrak{p}}$, where $\mathfrak{N}_{\mathfrak{p}}$ denotes the unique maximal m-subgroup of $\mathfrak{C}_{\mathfrak{p}}$, and thus the lattice-group-homomorphism $G \to \mathfrak{C}_{\mathfrak{p}} \to \mathfrak{C}_{\mathfrak{p}}/\mathfrak{N}_{\mathfrak{p}}$ of G onto the simple lattice-group $\mathfrak{C}_{\mathfrak{p}}/\mathfrak{N}_{\mathfrak{p}}$ does not map x onto the unit class. It is obvious that conversely if there is a lattice-group-homorphism of G onto a simple lattice-group not mapping an element x to the unit class, then x is not nilpotent; observe that a simple abelian lattice-group is linearly ordered. Thus the totality of nilpotent elements in G coincides with the intersection of all the kernels of (lattice-group-)homomorphic mappings of G onto simple lattice-groups.

The same remains true when G has the linearly ordered group of real number as an operator group, that is, when G is a vector-lattice, and so Yosida-Fukamiya's theorem is proved[1)2)].

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1) Yosida-Fukamiya, l. c., Theorem 1.

2) K. Yosida has pointed out that in case of an abelian lattice-group G its embedding into a direct product of linearly ordered groups may be obtained also from the following argument: Let G be additively written, and let x be a non-zero element in G. If G is not linearly ordered, then there exists a non-trivial m-subgroup (an ideal in Yosida's terminology) M not containing x. To see this, suppose $a \not\leqq b$, $a \not\geqq b$. Denote by M_1 the m-subgroup consisting of all the elements x such that $|x| \leqq n((b-a) \smile 0)$ for some n, where $|x| = (x \smile 0) - (x \frown 0)$. Similarly, let M_2 be the set of x such that $|x| \leqq n((a-b) \smile 0)$ for some n. Then at least one of M_1 and M_2 does not contain x and may be employed as our M.