11. An Abstract Integral, V.

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Introduction. S. Banach has introduced an integral which has no convergence properties, and which is defined for all bounded functions in $(0, 1)$. Evidently this class does not contain the class (L) of Lebesgue integrable functions. Since Banach integral is the integral without convergence properties, it will be desirable to define the Banach integral so that the class (B) of Banach integrable functions contains the class (L) and if Banach integral is pressed to have convergence properties of Lebesgue, then (B) reduces to (L). This is possible by the Jessen-Khintchine theorem.

In the case of abstract-integral, it is desirable to define such integral. For this purpose, we have introduced the abstract Banach integral for which above relation holds for the abstract-Lebesgue integrals in the third and fourth papers. This is given in § 1. In § 2 we define the second Banach integral such that above relation holds for abstract Riemann integral, § 3 contains a certain uniqueness theorem of above two integrals.

§ 4 contains that above consideration can be extended to the case where the value of the integral lies in a semi-vector lattice instead of real number field.

§ 1. Let E be a partially ordered linear space whose elements are denoted by $x, y, \dots$, and M be a set of elements $\alpha, \beta, \dots$. Now we shall consider a set of operations $T^\alpha x (\alpha \in M)$ which transforms E into the space of real numbers, and satisfies the following conditions.

(1.1) For every elements α, β of M and x, y of E, there exists a γ of M such that $T^\gamma(x+y) \leqq T^\alpha x + T^\beta y$.

(1.2) $\lambda T^\alpha x = T^\alpha(\lambda x)$, for any real number λ.

(1.3) If $x \leqq 0$, then $T^\alpha x \leqq 0$.

(1.4) For any element α of M, there exists an element e of M such that $T^\alpha e = 1$.

If we put

(1.5) $p(x) \equiv \text{g. l. b.} (T^\alpha x ; \alpha \in M)$[1),]

then we have

(1.6) $p(x+y) \leqq p(x) + p(y)$ for every x and y in E.

(1.7) $p(tx) = tp(x)$ for $t \geqq 0$.

Proof. For every $\varepsilon > 0$ and every x, y in E, we can find α and β in M such that

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1) g. l. b. $(T^\alpha x ; \alpha \in M)$ means the greatest lower bound of $T^\alpha x$ when α runs over M. For the least upper bound we use the similar notation.

$$T^{\alpha}x < p(x)+\varepsilon\,, \qquad T^{\beta}y < p(y)+\varepsilon$$

by the definition (1.6). By (1.1) there exists γ of M such that

$$T^{\gamma}(x+y) \leqq T^{\alpha}x + T^{\beta}y\,.$$

Consequently

$$p(x+y) \leqq p(x)+p(y)+2\varepsilon$$

and then letting $\varepsilon \to 0$, we get (1.6). The relation (1.7) is evident by (1.2) and (1.5).

By (1.6), (1.7) and the Banach's extension theorem, there exists a linear functional $f(x)$ on the space E such that $p(x) \geqq f(x)$.

We denote $f(x)$ by $\int x$, then this integral has the following properties;

(1.8) $\int(ax+by)=a\int x+b\int y$, where a and b are real numbers.

(1.9) $x \geqq 0$ implies $\int x \geqq 0$.

(1.10) $\int e=1$.

Proof. By the linearlity of $f(x)$, (1.8) is evident. And $x \geqq 0$ implies $-x \leqq 0$. By (1.3) $T^{\alpha}(-x) \leqq 0$, and then $p(-x) \leqq 0$. Therefore $\int(-x) \leqq p(-x) \leqq 0$, and then $\int x = -\int(-x) \geqq 0$. This proves (1.9).

Now $p(-e)=\text{g. l. b.}\left(T^{\alpha}(-e);\, \alpha\in M\right) = -\text{l. u. b.}\left(T^{\alpha}(e);\, \alpha\in M\right) = -1$, and then

$$p(e)=\text{g. l. b.}\,(T^{\alpha}e;\, \alpha\in M)=1\,.$$

Consequently

$$1=p(e) \geqq \int e = -\int(-e) \geqq -p(-e)=1\,,$$

that is $\int e=1$, which is (1.10).

§ **2.** Let E_1 be a vector lattice such that

(2.1) there exists an elements e which we call unit such that for every element x of E_1 there exists a positive number p such that $-pe \leqq x \leqq pe$.

Let M_1 be an additive group. We shall consider a set of operations $T^{\alpha}x(\alpha\in M_1)$ which transforms E_1 into E_1, where x and α are respectively the elements of E_1 and M_1.

We suppose that

(2.2) $T^{\alpha}e=e$.

(2.3) $T^{\alpha}(x+y)=T^{\alpha}x+T^{\alpha}y$.

(2.4) $T^{\alpha+\beta}x=T^{\alpha}(T^{\beta}x)$.

(2.5) For any real number λ, $\lambda T^{\alpha}x=T^{\alpha}(\lambda x)$.

(2.6) $x \leqq 0$ implies $T^{\alpha}x \leqq 0$.

(2.7) for any $x\in E_1$, there exists C such that

$$|T^{\alpha}x| \leqq C|x| \text{ for any } \alpha \text{ in } M_1.$$

If we put

(2.8) $\quad \wedge(\lambda;\, \alpha_1, \cdots, \alpha_m) = \text{g. l. b.}\left(\lambda;\, \frac{1}{m}\sum_{k=1}^{m} T^{\alpha_k}x \leqq \lambda e\right),$

(2.9) $\quad p(x) = \text{g. l. b.}\left(\wedge(\lambda;\, \alpha_1, \cdots, \alpha_m);\ m \in I,\ \alpha_i \in M_1\right),$

where I denotes the set of all integers, then we have

(2.10) $\quad p(x+y) \leqq p(x) + p(y),$

(2.11) $\quad p(tx) = tp(x)$, where $t \geqq 0$.

Proof. (2.11) is evident by (2.5), (2.8) and (2.9).

By (2.8) and (2.9) there exist $\alpha_1, \alpha_2, \cdots, \alpha_u;\ \beta_1, \beta_2, \cdots, \beta_v$ in M such that $\wedge(x;\, \alpha_1, \alpha_2, \cdots, \alpha_u) < p(x) + \epsilon$ and $\wedge(y;\, \beta_1, \beta_2, \cdots, \beta_v) < p(y) + \epsilon$.

Now $\frac{1}{u}\sum_{i=1}^{u} T^{\alpha_i}x \leqq \lambda e$ implies $\frac{1}{u}\sum_{i=1}^{u} T^{\alpha_i}\cdot T^{\beta_k}x \leqq \lambda e$ by (2.2), (2.3) and also $\frac{1}{uv}\sum_{i=1}^{n}\sum_{k=1}^{v} T^{\alpha_i+\beta_k}x \leqq \lambda e$ by (2.4)

Similarly $\frac{1}{v}\sum_{k=1}^{v} T^{\beta_k}y \leqq \mu e$ implies $\frac{1}{uv}\sum_{i=1}^{u}\sum_{k=1}^{v} T^{\alpha_i+\beta_k}y \leqq \mu e$. Thus we have $\frac{1}{uv}\sum_{i=1}^{u}\sum_{k=1}^{v} T^{\alpha_i+\beta_k}(x+y) \leqq (\lambda+\mu)e$. Consequently $\wedge\left(x+y;\, \alpha_i+\beta_k, \begin{matrix} i=1, 2, 3, \cdots, u \\ k=1, 2, 3, \cdots, v \end{matrix}\right) \leqq \wedge(x;\, \alpha_1, \alpha_2, \cdots, \alpha_u) + \wedge(y;\, \beta_1, \beta_2, \cdots, \beta_v)$ and then

$$p(x+y) \leqq p(x) + \epsilon + q(x) + \epsilon = p(x) + q(x) + 2\epsilon\,.$$

Thus $p(x+y) \leqq p(x) + p(y)$, which is (2.10).

As in §2 there exists a linear functional $f(x) = \int x$ such that $f(x) \leqq p(x)$. Then we have

(2.12) $\quad \int(ax+by) = a\int x + b\int y$, where a, b are real.

(2.13) $\quad \int e = 1.$

(2.14) $\quad x \geqq 0$ implies $\int x \geqq 0$.

(2.15) $\quad \int T^{\alpha}x = \int x$ for all $\alpha \in M$.

Proof. (2.12)–(2.14) is proved similarly as in §2. It remains to prove (2.15). If we put $\alpha_k = (k-1)\alpha$ where k is a positive integer, then

$$\frac{1}{m}\sum T^{\alpha_k}(T^{\alpha}x - x) = \frac{1}{m}\sum\left\{T^{\alpha_k+\alpha}x - T^{\alpha_k}x\right\}$$

$$= \frac{1}{m}\sum\left\{T^{\alpha}x - T^{(k-1)\alpha}x\right\} = \frac{1}{m}\sum\left\{T^{m\alpha}x - x\right\}$$

which tends to 0 as $m \to \infty$ by (2.7). Thus $p(T^{\alpha}x - x) = 0$. Similarly $p(x - T^{\alpha}x) = 0$, and then

$$p(T^{\alpha}x - x) \geqq \int(T^{\alpha}x - x) = \int T^{\alpha}x - \int x \geqq -p(x - T^{\alpha}x)$$

which gives $\int T^{a}x=\int x$, that is, (2.16) is proved.

§ **3.** In order to prove a unicity theorem we put $q(x)\equiv -p(-x)$, then we have

(3.1) $q(tx)=tq(x)$ for all $t\geqq 0$

(3.2) $q(x+y)\geqq q(x)+q(y)$.

Proof. Evident by

$$q(tx)=-p(-tx)=-tp(-x)=tq(x)$$

and

$$q(x+y)=-p(-x-y)\geqq -p(-x)-p(-y)=q(x)+q(y)\,.$$

We have also

(3.3) $p(0)=q(0)=0$,

(3.4) $p(x)\geqq q(x)$.

Proof. Putting $t=0$ in (2.1) and (3.1), we get (3.3). (3.4) is evident by

$$p(x)-q(x)=p(x)+p(-x)\geqq p(x-x)=p(0)=0\,.$$

Let $\bar{E}_1$ be a subset of E_1 such that $\bar{E}_1\equiv E_1\big(x\,;\,p(x)=q(x)\big)$. As $\int x$ is defined by $f(x)=\int x$, and the uniqueness of $f(x)$ hard to understand, also the uniqueness of $\int x$. But

The "integral" having above four properties (2.12)–(2.15) *coincides with* $\int x$ *in* $\bar{E}_1$.

Proof. If $f(x)$ has the properties (2.12)–(2.15), then for any x in $\bar{E}_1$ $\frac{1}{m}\sum T^{a_k}x\leqq \lambda e$ implies $f(x)\leqq \lambda e$, that is $f(x)\leqq p(x)$. We have

$$p(x)\geqq f(x)=-f(-x)\geqq -p(-x)=q(x)\,.$$

However

$$p(x)\geqq \int x=-\int -x\geqq -p(-x)=q(x)\,,$$

accordingly

$$p(x)=q(x)=\int x=f(x)\quad\text{for all}\quad x\in\bar{E}_1\,.$$

We can prove the similar theorem concerning the integral in § 1.

§ **4.** Let E and E_2 be partially ordered linear spaces such that

(4.1) In E_2, "join" is defined and E_2 is (conditionally) join-complete, that is,

$$a_\alpha\geqq a\quad\text{for all}\quad \alpha\in A\quad\text{implies}\quad \bigwedge(a_\alpha\,;\,\alpha\in A)\ \text{exists.}$$

Then we can prove the analogue of the Banach's extension theorem.

If $f(x)$ is an operation which transforms E into E_2 such that

(4.2) $p(x+y)\leqq p(x)+p(y)$,

(4.4) $p(tx)=tp(x)$ for all $t\geqq 0$,

then there is a non-negative linear functional $f(x)$ such that

(4.4) $f(x) \leqq p(x)$

where "$f(x)$ is non-negative" means that

(4.5) $x \leqq 0$ implies $f(x) \leqq 0$.

Proof is done similarly as the Banach's theorem. By this theorem we can define two integrals of x in E with value in E_2, one being similar as § 1 and the other being similar as in § 2.