29. On the Behaviour of an Inverse Function of a Meromorphic Function at its Transcendental Singular Point, III.

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1. Nevanlinna's fundamental theorems.

Let $w=w(z)=f(z)$ be a meromorphic function for $|z|<\infty$ and $z=\varphi(w)$ be its inverse function. Let K be the Riemann sphere of diameter 1, which touches the w-plane at $w=0$ and $[a, b]=\dfrac{|a-b|}{\sqrt{(1+|a|^2)(1+|b|^2)}}$.

A δ-neighbourhood U of w_0 is the connected part of the Riemann surface F of $\varphi(w)$, which lies in $[w, w_0]<\delta$ and has w_0 as an inner point or as a boundary point. Let U correspond to $\varDelta$ on the z-plane, then $[f(z), w_0]<\delta$ in $\varDelta$ and $[f(z), w_0]=\delta$ on the boundary of $\varDelta$. We assume that $\varDelta$ extends to infinity. Let z_0 be a point on the z-plane and $\varDelta_r$, θ_r be the common part of $\varDelta$ and $|z-z_0|\leqq r$ and $|z-z_0|=r$ respectively. We put $A(r, w; \varDelta)=$ the area on K, which is covered by $w=f(z)$, when z varies in $\varDelta_r$, $S(r, w; \varDelta)=\dfrac{A(r, w; \varDelta)}{\pi\delta^2}$, where $\pi\delta^2$ is the area of $[w, w_0]\leqq\delta$ on K, $n(r, a, w; \varDelta)=$ the number of zero points of $f(z)-a$ in $\varDelta_r$, where $[a, w_0]<\delta$.

$$N(r, a, w; \varDelta)=\int_{r_0}^{r}\frac{n(r, a, w; \varDelta)}{r}\,dr\,,$$

$$m(r, a, w; \varDelta)=\frac{1}{2\pi}\int_{\theta_r}\log\frac{1}{[w(re^{i\varphi}), a]}\,d\varphi\,,$$

$$T(r, a, w; \varDelta)=N(r, a, w; \varDelta)+m(r, a, w; \varDelta)\,,$$

$L(r)=$ the total length of the curve on K, which corresponds to θ_r. Then we have the following theorem[1], which corresponds to *Nevanlinna's first fundamental theorem.*

Theorem I. $\quad T(r, a, w; \varDelta)=T(r, w; \varDelta)+O\left(\int_{r_0}^{r}\frac{L(r)}{r}\,dr\right),$

where $$T(r, w; \varDelta)=\int_{r_0}^{r}\frac{S(r, w; \varDelta)}{r}\,dr\,.$$

We will call $T(r, w; \varDelta)$ the characteristic function of $f(z)$ in $\varDelta$ and

1) C. f. K. Kunugui: Une généralisation des théorèmes de MM. Picard-Nevanlinna sur les fonctions méromorphes. Proc. **17** (1941), 283–289.

Y. Tumura: Sur le problème de M. Kunugui. Proc. **17** (1941), 289–295.

Mr. Tumura obtained the same result as Theorem 1, but he informed me that he found a mistake in his proof and will publish a revised proof in this proceedings.

$\overline{\lim_{r\to\infty}} \frac{\log T(r, w; \Delta)}{\log r} = \rho$ the order of $f(z)$ in Δ. We will first prove the following theorem.

Theorem II. Let $U(w)$ be a linear transformation, which makes $[w, w_0] < \delta$ invariant, then $S(r, w; \Delta) - S(r, U(w); \Delta) = O(L(r))$.

Proof. Let Γ_r' be the whole boundary of Δ_r and $\Gamma_r = \theta_r + \gamma_r$ and a, b be any two points in $[w, w_0] \leqq \delta_1 < \delta$, then

$$\frac{1}{2\pi}\int_{\theta_r} \frac{d}{dr} \log \frac{[w, b]}{[w, a]} d\varphi = \frac{1}{2\pi}\int_{\theta_r} \frac{d}{dr} \log \frac{|w-b|}{|w-a|} d\varphi$$

$$= \frac{1}{2\pi r}\int_{\Gamma_r} d \arg \frac{w-b}{w-a} - \frac{1}{2\pi r}\int_{\gamma_r} d \arg \frac{w-b}{w-a}$$

$$= \frac{n(r, b, w; \Delta) - n(r, a, w; \Delta)}{r} - \frac{1}{2\pi r}\int_{\gamma_r} d \arg \frac{w-b}{w-a}. \quad (1)$$

Since a, b, lie in $[w, w_0] \leqq \delta_1$, we have easily

$$\left| \frac{1}{2\pi r}\int_{\gamma_r} d \arg \frac{w-b}{w-a} \right| \leqq K(\delta_1) \frac{L(r)}{r},$$

where $K(\delta_1)$ depends on δ_1 only. Hence

$$I = \frac{n(r, a, w; \Delta)}{r} + \frac{1}{2\pi}\int_{\theta_r} \frac{d}{dr} \log \frac{1}{[w, a]} d\varphi$$

$$= \frac{n(r, b, w; \Delta)}{r} + \frac{1}{2\pi}\int_{\theta_r} \frac{d}{dr} \log \frac{1}{[w, b]} d\varphi + O\left(\frac{L(r)}{r}\right). \quad (2)$$

Let $d\omega(b)$ the surface element on K at b, then since $\pi\delta_1^2$ is the area of $[w, w_0] \leqq \delta_1$, taking the integral mean over $[w, w_0] \leqq \delta_1$, we have

$$I = \frac{S_1(r, w; \Delta)}{r} + \frac{1}{2\pi^2\delta_1^2}\int_{\theta_r} d\varphi \int_{[b, w_0] \leqq \delta_1} \frac{d}{dr} \log \frac{1}{[w, b]} d\omega(b)$$

$$+ O\left(\frac{L(r)}{r}\right), \quad (3)$$

where $S_1(r, w; \Delta) = \frac{A_1(r, w; \Delta)}{\pi\delta_1^2}$, $A_1(r, w; \Delta)$ being the area on K over $[w, w_0] \leqq \delta_1$, which is covered by $w = f(z)$, when z varies in Δ_r. By Ahlfors' theorem,

$$S(r, w; \Delta) - S_1(r, w; \Delta) = O(L(r)), \quad (4)$$

so that

$$\frac{n(r, a, w; \Delta)}{r} + \frac{1}{2\pi}\int_{\theta_r} \frac{d}{dr} \log \frac{1}{[w, a]} d\varphi$$

$$= \frac{S(r, w; \Delta)}{r} + \frac{1}{2\pi^2\delta_1^2}\int_{\theta_r} d\varphi \int_{[b, w_0] \leqq \delta_1} \frac{d}{dr} \log \frac{1}{[w, b]} d\omega(b)$$

$$+ O\left(\frac{L(r)}{r}\right). \quad (5)$$

We have a similar expression for $U(w)$. Since $n(r, a, w; \Delta) = n\big(r, U(a), U(w); \Delta\big)$, we have

$$\frac{1}{2\pi}\int_{\theta_r} \frac{d}{dr} \log \frac{[U(w), U(a)]}{[w, a]} d\varphi = \frac{S(r, w; \Delta) - S(r, U(w); \Delta)}{r}$$

$$+ \frac{1}{2\pi^2\delta_1^2}\int_{\theta_r} d\varphi \int_{[b, w_0]\leqq \delta_1} \frac{d}{dr} \log \frac{[U(w), b]}{[w, b]} d\omega(b) + O\Big(\frac{L(r)}{r}\Big). \quad (6)$$

By means of Dinghas' theorem[1], we can prove, if $[b, w_0] \leqq \delta_1 < \delta$,

$$\left| \frac{d}{dr} \log \frac{[U(w), U(a)]}{[w, a]} \right| \leqq K \frac{|w'|}{1+|w|^2},$$

$$\left| \frac{d}{dr} \log \frac{[U(w), b]}{[w, b]} \right| \leqq K \frac{|w'|}{1+|w|^2},$$

where K is a constant. Since $L(r) = r\int_{\theta_r} \frac{|w'|}{1+|w|^2} d\varphi$, we have

$$S(r, w; \Delta) - S\big(r, U(w); \Delta\big) = O\big(L(r)\big). \qquad \text{q. e. d.}$$

Proof of Theorem I. Let θ_r consist of circular arcs whose end points are $re^{i\theta_1}$, $re^{i\theta_2}$ and let $\theta(r) = \sum \big(\theta_2(r) - \theta_1(r)\big)$. We put $w_1 = w(re^{i\theta_1})$, $w_2 = w(re^{i\theta_2})$, then $[w_1, w_0] = [w_2, w_0] = \delta$. Let a be a point in $[w, w_0] \leqq \delta_1 < \delta$, then by (1),

$$\frac{d}{dr} m(r, w_0, w; \Delta) - \frac{d}{dr} m(r, a, w; \Delta)$$

$$= \frac{1}{2\pi} \sum \Big(\log \frac{[w_2, a]}{[w_2, w_0]} \frac{d\theta_2}{dr} - \log \frac{[w_1, a]}{[w_1, w_0]} \frac{d\theta_1}{dr}\Big)$$

$$+ \frac{n(r, a, w; \Delta) - n(r, w_0, w; \Delta)}{r} + O\Big(\frac{L(r)}{r}\Big),$$

so that

$$T(r, w_0, w; \Delta) = T(r, a, w; \Delta) = \frac{1}{2\pi} \sum \int_{r_0}^{r} \Big(\log \frac{[w_2, a]}{\delta} \frac{d\theta_2}{dr}$$

$$- \log \frac{[w_1, a]}{\delta} \frac{d\theta_1}{dr}\Big) dr + O\Big(\int_{r_0}^{r} \frac{L(r)}{r} dr\Big). \quad (7)$$

Multiplying $d\omega(a)$ and taking the integral mean over $[w, w_0] \leqq \delta_1 < \delta$, we have

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1) A. Dinghas: Zur Invarianz der Shimizu-Ahlforsschen Charakteristik. Math. Z. **45**, 25–28.

$$T(r, w_0, w; \Delta)=\frac{1}{\pi\delta_1^2}\int_{[a, w_0]\leq\delta_1} T(r, a, w; \Delta)\, d\omega(a)$$

$$+\frac{1}{2\pi^2\delta_1^2}\int_{r_0}^{r} dr\sum\int_{[a, w_0]\leq\delta_1}\left(\log\frac{[w_2, a]}{\delta}\frac{d\theta_2}{dr}-\log\frac{[w_1, a]}{\delta}\frac{d\theta_1}{dr}\right)d\omega(a)$$

$$+O\left(\int_{r_0}^{r}\frac{L(r)}{r}dr\right). \tag{8}$$

We see easily that by (4)

$$\frac{1}{\pi\delta_1^2}\int_{[a, w_0]\leq\delta_1} T(r, a, w; \Delta)\, d\omega(a)=\int_{r_0}^{r}\frac{S(r, w; \Delta)}{r}dr+O\left(\int_{r_0}^{r}\frac{L(r)}{r}dr\right).$$

Since w_1 and w_2 lie on $[w, w_0]=\delta$,

$$\int_{[a, w_0]\leq\delta_1}\log\frac{[w_1, a]}{\delta}d\omega(a)=\int_{[a, w_0]\leq\delta_1}\log\frac{[w_2, a]}{\delta}d\omega(a)=A=\text{const.},$$

hence the second term of (8) becomes

$$\frac{A}{2\pi^2\delta_1^2}\int_{r_0}^{r}\frac{d}{dr}\sum\left(\theta_2(r)-\theta_1(r)\right)dr=\frac{A}{2\pi^2\delta_1^2}\left(\theta(r)-\theta(r_0)\right)=O(1).$$

Hence

$$T(r, w_0, w; \Delta)=\int_{r_0}^{r}\frac{S(r, w; \Delta)}{r}dr+O\left(\int_{r_0}^{r}\frac{L(r)}{r}dr\right). \tag{9}$$

Let a be a point in $[w, w_0]<\delta$ and $U(w)$ be a linear transformation which makes $[w, w_0]<\delta$ invariant and carries a to w_0, so that $w_0=U(a)$, then

$$T\left(r, U(a), U(w); \Delta\right)=T\left(r, w_0, U(w); \Delta\right)$$

$$=N\left(r, U(a), U(w); \Delta\right)+\frac{1}{2\pi}\int_{\theta_r}\log\frac{1}{[U(w), U(a)]}d\varphi$$

$$=N(r, a, w; \Delta)+\frac{1}{2\pi}\int_{\theta_r}\log\frac{1}{[w, a]}d\varphi+O(1)$$

$$=T(r, a, w; \Delta)+O(1). \tag{10}$$

Hence from (9), (10) and Theorem II, we have

$$T(r, a, w; \Delta)=T\left(r, w_0, U(w); \Delta\right)+O(1)$$

$$=\int_{r_0}^{r}\frac{S(r, U(w); \Delta)}{r}dr+O\left(\int_{r_0}^{r}\frac{L(r)}{r}dr\right)$$

$$=\int_{r_0}^{r}\frac{S(r, w; \Delta)}{r}dr+O\left(\int_{r_0}^{r}\frac{L(r)}{r}dr\right). \qquad \text{q. e. d.}$$

Remark I. Let D be a domain on K, which is bounded by an analytic Jordan curve C and D correspond to Δ on the z-plane by $w=w(z)=f(z)$. We map D conformally on $[v, v_0]<\delta$ by $w=\psi(v)$, then $w(z)$ becomes $v(z)$. Let $L_1(r)$, $L(r)$ be the length of the curve on K, which correspond to θ_r by $v=v(z)$, $w=w(z)$ respectively, then $L_1(r)=O\big(L(r)\big)$. By Theorem I, for any two points α, β in $[v, v_0]\leqq \delta_1<\delta$,

$$T(r, \alpha, v; \Delta)=T(r, \beta, v; \Delta)+O\Big(\int_{r_0}^{r}\frac{L_1(r)}{r}dr\Big)$$
$$=T(r, \beta, v; \Delta)+O\Big(\int_{r_0}^{r}\frac{L(r)}{r}dr\Big).$$

Since $T(r, \alpha, v; \Delta)=T(r, a, w; \Delta)+O(1)$, where $a=\psi(\alpha)$, we have for any two points a, b in $D_1\subset D$,

$$T(r, a, w; \Delta)=T(r, b, w; \Delta)+O\Big(\int_{r_0}^{r}\frac{L(r)}{r}dr\Big).$$

Multiplying $d\omega(b)$ and taking the integral mean over $D_1\,(\subset D)$, we have

$$T(r, a, w; \Delta)=\int_{r_0}^{r}\frac{S(r, w; \Delta)}{r}dr+O\Big(\int_{r_0}^{r}\frac{L(r)}{r}dr\Big).$$

Hence *Theorem I holds, if $[w, w_0]<\delta$ is replaced by any domain bounded by an analytic Jordan curve on K.*

II. Since

$$A(r, w; \Delta)=\iint_{\Delta_r}\frac{|w'|^2}{(1+|w|^2)^2}rdrd\varphi, \qquad L(r)=r\int_{\theta_r}\frac{|w'|}{1+|w|^2}d\varphi,$$

we have

$$[L(r)]^2\leqq 2\pi r\int_{\theta_r}\frac{|w'|^2}{(1+|w|^2)^2}rd\varphi=2\pi r\frac{dA}{dr},$$

$$\int_{r_0}^{r}\frac{[L(r)]^2}{r}dr\leqq 2\pi A(r, w; \Delta)=O\big(T(2r, w; \Delta)\big),$$

so that

$$\int_{r_0}^{r}\frac{L(r)}{r}dr\leqq\sqrt{\log r\int_{r_0}^{r}\frac{[L(r)]^2}{r}dr}=O\big(\sqrt{T(2r, w; \Delta)\log r}\big). \quad (11)$$

Dinghas[1)] proved that

$$\int_{r_0}^{r}\frac{L(r)}{r}dr=O\big(\sqrt{T(r, w; \Delta)}\log T(r, w; \Delta)\big),$$

except certain intervals I_n, such that $\sum_n\int_{I_n}d\log r<\infty$.

1) A. Dinghas: Eine Bemerkung zur Ahlforsschen Theorie der Überlagerungsflächen. Math. Z. **44**.

III. In my former paper[1] I have proved that,

$$(q-1)S(r,w;\Delta)\leqq\sum_{i=1}^{q}n(r,a_i,w;\Delta)+\Lambda(r)+O\big(L(r)\big)\quad([a_i,w_0]<\delta),$$

where $\Lambda(r)$ is *the number of holes in* Δ_r, which is $\leqq S(r,w;\Delta)+O\big(L(r)\big)$. Hence putting $\Gamma(r)=\int_{r_0}^{r}\frac{\Lambda(r)}{r}dr$, we have

Theorem III. For any $q(\geqq 2)$ *points* a_i *in* $[w,w_0]<\delta$,

$$(q-1)T(r,w;\Delta)\leqq\sum_{i=1}^{q}N(r,a_i,w;\Delta)+\Gamma(r)+O\Big(\int_{r_0}^{r}\frac{L(r)}{r}dr\Big),\tag{12}$$

where $$\Gamma(r)\leqq T(r,w;\Delta)+O\Big(\int_{r_0}^{r}\frac{L(r)}{r}dr\Big).$$

This corresponds to *Nevanlinna's second fundamental theorem.*

From (11), (12) and Theorem I, we have the following theorem, which corresponds to Borel's theorem.

Theorem IV. Let $f(z)$ *be a meromorphic function of finite order* ρ *in* Δ *and* $r_n(a)$ *be the absolute values of the zero points of* $f(z)-a$ *in* Δ, *then* $\sum_{n=1}^{\infty}\frac{1}{[r_n(a)]^{\rho+\varepsilon}}$ $(\varepsilon>0)$ *is convergent for all* a *in* $[w,w_0]<\delta$ *and* $\sum_{n=1}^{\infty}\frac{1}{[r_n(a)]^{\rho-\varepsilon}}$ *is divergent, except at most two values of* a *in* $[w,w_0]<\delta$. *If* $\overline{\lim_{r\to\infty}}\frac{\log\Gamma(r)}{\log r}<\rho$, *then* $\sum_{n=1}^{\infty}\frac{1}{[r_n(a)]^{\rho-\varepsilon}}$ *is divergent except at most one value of* a.

2. Ahlfors' theorem on the number of asymptotic values.

Let w_0 be a transcendental singularity of an inverse function $z=\varphi(w)$ of a meromorphic function $w=f(z)$ for $|z|<\infty$ and w_0 is an accessible boundary point of the Riemann surface F of $\varphi(w)$ and a δ-neighbourhood U of w_0 correspond to Δ on the z-plane. From the accessibility of w_0, Δ contains a curve C extending to infinity, such that $[f(z),w_0]$ tends to zero, when z tends to infinity along C. By Iversen, w_0 is called a direct transcendental singularity, if $f(z)-w_0$ has no zero points in Δ. Ahlfors[2] proved that if $f(z)$ is of finite order ρ, then the number n of direct transcendental singularities is $\leqq 2\rho$, if $n\geqq 2$. We will show that if the number of zero points of $f(z)-w_0$ in Δ is not so large, then the number n of such singularities is $\leqq 2\rho$, if $n\geqq 2$. For this purpose, we will introduce a new notion "quasi-direct transcendental singularity" as follows. Now the boundary of Δ consists of two classes of curves. Namely the ones which extend to infinity and the others which are closed curves and form holes of Δ. We add all such holes to Δ and the resulting simply connected domain be denoted

1) M. Tsuji: On the behaviour of an inverse function of a meromorphic function at its transcendental singular point. Proc. **17** (1941), 414–417.

2) L. Ahlfors: Über die asymptotischen Wert der meromorphen Funktionen endlicher Ordnung. Acta Acad. Aboensis, Math. et Phys. **6** (1932).

by $\bar{\varDelta}$. If $\varDelta$ has boundary curves which extend to infinity, let Γ be the outermost such curve and the simply connected domain bounded by Γ be denoted by $\bar{\bar{\varDelta}}$. If there is no such curve, we take the whole z-plane as $\bar{\bar{\varDelta}}$. We also denote the total length of the common part of $|z-z_0|=r$ and $\varDelta$, $\bar{\varDelta}$, $\bar{\bar{\varDelta}}$ by $r\theta(r)$, $r\bar{\theta}(r)$, $r\bar{\bar{\theta}}(r)$ respectively.

Let $n(r)$ be the number of zero points of $f(z)-w_0$ in the common part of $\varDelta$ and $|z-z_0|\leqq r$. We will call w_0 a *quasi-direct transcendental singularity*, if for any choice of z_0 and U,

$$n(r)\leqq K\int_{r_0}^{r}\frac{dr}{r\bar{\theta}(r)}\,, \tag{13}$$

where K is independent of r, but may depend on z_0 and U.

Then the following theorem holds.

Theorem V. Let $f(z)$ be a meromorphic function of finite order ρ for $|z|<\infty$, then the number n of quasi-direct transcendental singularities of $\varphi(w)$ is $\leqq 2\rho$, if $n\geqq 2$.

To prove Theorem V, we first prove the following theorem.

Theorem VI. Let $0<|a_n|<1$ and $n(r)$ be the number of $a_n(|a_n|\leqq r<1)$ and $F(z)=\prod_{n=1}^{\infty}\frac{\bar{a}_n}{|a_n|}\frac{a_n-z}{1-\bar{a}_n z}\ (|z|<1)$. If $n(r)\leqq K\log\frac{1}{1-r}$ (K=const.), then there exists a sequence $r_n\to 1$, such that $\underset{|z|=r_n}{\mathrm{Min.}}\ |F(z)|\geqq\delta>0$ (δ=const.).

To prove Theorem V, let w_0 be a quasi-direct transcendental singularity, which we assume $w_0=\infty$. Let U be a δ-neighbourhood of w_0, which corresponds to $\varDelta$ on the z-plane and z_n be the poles of $f(z)$ in $\varDelta$, which satisfy (13). We map $\bar{\varDelta}$ on $|\zeta|<1$ and let z_n become ζ_n in $|\zeta|<1$, then by Ahlfors' Verzerrungssatz, ζ_n satisfy the condition of theorem VI. We put $G(z)=g(\zeta)=\prod_{n=1}^{\infty}\frac{\bar{\zeta}_n}{\zeta_n}\frac{\zeta_n-\zeta}{1-\bar{\zeta}_n\zeta}$, then $|G(z)|\leqq 1$ in and on the boundary of $\varDelta$ and $G(z_n)=0$, so that $F(z)=G(z)f(z)$ is regular in $\varDelta$ and $|F(z)|\leqq|f(z)|$ in $\varDelta$. $F(z)$ is unbounded in $\varDelta$. For, if $F(z)$ is bounded in $\varDelta$ and $|F(z)|\leqq K$, then $|G(z)|\leqq\frac{K}{|f(z)|}$. By the hypothesis, $\varDelta$ contains a curve C, such that $f(z)\to\infty$ along C, so that $G(z)\to 0$ along C, which contradicts Theorem VI. Hence $F(z)$ is unbounded in $\varDelta$. From this, we proceed exactly in the same way as Ahlfors' proof and complete the proof.

If we apply Theorem I, we can prove the following extension.

Theorem VII. Let a δ-neighbourhood U of an accessible singularity of an inverse function $z=\varphi(w)$ of a meromorphic function $w=f(z)$ contain n quasi-direct transcendental singularities and U correspond to $\varDelta$ on the z-plane. If $f(z)$ is of finite order ρ in $\varDelta$, then if $n\geqq 2$,

$$n\leqq\frac{\rho}{\pi}\Big/\varlimsup_{r\to\infty}\frac{1}{\log r}\int_{r_0}^{r}\frac{dr}{r\bar{\bar{\theta}}(r)}\,. \tag{14}$$

If $\varDelta$ has a boundary curve extending to infinity, then (14) *holds without the restriction,* $n \geqq 2$.

It is easily seen that Theorem V contains the following Valiron's theorem[1] as a special case.

Corollary. *If* $T(r)=O\big((\log r)^2\big)$, *then there is at most one asymptotic value.*

Mr. Tumura[2] proved that $T(r)=O\big((\log r)^2\big)$ can be replaced by $\varliminf_{r\to\infty} \dfrac{T(r)}{(\log r)^2} < \infty$.

The full detail of the proof will appear in the Japanese Journal of Mathematics, **18.**

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1) G. Valiron: Sur les valeurs asymptotiques de quelques fonctions méromorphes. Rendiconti Circolo mat. di Palermo. **46** (1925).

Sur le nombre des singularités transcendantes des fonctions inverse d'une classe d'algébroide. C. R. **200** (1936).

2) Y. Tumura: Sur le théorème de M. Valiron et les singularités transcendant indirectement critiques. Proc. **17** (1941), 65–69.