36. On a Theorem of F. and M. Riesz.

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(Comm. by S. KAKEYA, M.I.A., April 13, 1942.)

1. Let D be a domain on the w-plane, bounded by a rectifiable curve Γ and we map D conformally on $|z|<1$, then F. and M. Riesz[1] proved that a null set on $|z|=1$ corresponds to a null set on Γ and a null set on Γ corresponds to a null set on $|z|=1$, where a set is called a null set, if its measure is zero. We will prove an analogous theorem, when D is a domain on a minimal surface, bounded by a rectifiable curve.

Let Γ be a rectifiable curve in an m-dimensional space, then it is proved by Radó, Douglas and Courant that there exists a minimal surface S through Γ.

Let S be defined by a vector $\mathfrak{x}=\mathfrak{x}(z)=\big(x_1(z), \dots, x_m(z)\big)$ $(z=u+iv=re^{i\theta})$, where the components $x_k(z)$ $(k=1, \dots, m)$ are continuous in $|z|\leqq 1$ and harmonic in $|z|<1$ and $\mathfrak{x}=\mathfrak{x}(e^{i\theta})$ maps $|z|=1$ continuously and monotonically on Γ and if we put

$$E=\sum_{k=1}^{m}\left(\frac{\partial x_k}{\partial u}\right)^2, \quad F=\sum_{k=1}^{m}\frac{\partial x_k}{\partial u}\cdot\frac{\partial x_k}{\partial v}, \quad G=\sum_{k=1}^{m}\left(\frac{\partial x_k}{\partial v}\right)^2,$$

then

$$E=G, \quad F=0 \quad \text{in } |z|<1. \tag{1}$$

Let ds be the line element on S, then

$$ds^2=\sum_{k=1}^{m} dx_k^2=E(du^2+dv^2)=E(dr^2+r^2d\theta^2), \tag{2}$$

so that

$$E=E(z)=\frac{1}{r^2}\sum_{k=1}^{m}\left(\frac{\partial x_k}{\partial \theta}\right)^2.$$

Put $x_k=\Re\big(f_k(z)\big)$, where $f_k(z)$ are regular in $|z|<1$, then

$$E=\frac{1}{2}(E+G)=\frac{1}{2}\sum_{k=1}^{m}\left(\left(\frac{\partial x_k}{\partial u}\right)^2+\left(\frac{\partial x_k}{\partial v}\right)^2\right)$$

$$=\frac{1}{2}\sum_{k=1}^{m}\left(\frac{\partial x_k}{\partial u}+i\frac{\partial x_k}{\partial v}\right)\left(\frac{\partial x_k}{\partial u}-i\frac{\partial x_k}{\partial v}\right)=\frac{1}{2}\sum_{k=1}^{m}|f_k'(z)|^2. \tag{3}$$

We will prove the following theorem.

Theorem I. Let S be a minimal surface in an m-dimensional space, bounded by a rectifiable curve Γ and $\mathfrak{x}=\mathfrak{x}(z)$ map S on $|z|\leqq 1$, then a null set on $|z|=1$ corresponds to a null set on Γ and a null set on Γ corresponds to a null set on $|z|=1$.

1) F. and M. Riesz: Über die Randwerte einer analytischen Funktion. Quatrième congres des mathématiciens scandinaves à Stockholm, 1916.

2. *Proof of Theorem I.* Let L be the length of Γ and $|z|=r<1$ corresponds to Γ_r on S and $L(r)$ be its length, then Radó[1] proved that $L(r)$ is an increasing function of r and

$$L(r) \leqq L\,, \quad \lim_{r\to 1} L(r)=L\,. \tag{4}$$

Since $L(r)=\int_0^{2\pi} r\sqrt{E(re^{i\theta})}\,d\theta$, we have from (3) and (4),

$$\int_0^{2\pi} r\,|f_k'(re^{i\theta})|\,d\theta \leqq \sqrt{2}\int_0^{2\pi} r\sqrt{E(re^{i\theta})}\,d\theta=\sqrt{2}\,L(r) \leqq \sqrt{2}\,L\,.$$

Hence by F. Riesz' theorem[2], $f_k(z)$ and hence $x_k(z)$ are absolutely continuous on $|z|=1$ and $\lim\limits_{z\to e^{i\theta}} f_k'(z)=f_k'(e^{i\theta})$ exist almost everywhere on $|z|=1$, when z tends to $e^{i\theta}$ non-tangentially to $|z|=1$ and $f_k'(e^{i\theta}) \neq 0$ almost everywhere, if $f_k(z) \not\equiv$ const. Since

$$E(e^{i\theta})=\lim_{z\to e^{i\theta}} E(z)=\lim_{z\to e^{i\theta}} \sum_{k=1}^{m}\left(\frac{\partial x_k(z)}{\partial \psi}\right)^2=\frac{1}{2}\lim_{z\to e^{i\theta}}\sum_{k=1}^{m}|f_k'(z)|^2$$

$$=\frac{1}{2}\sum_{k=1}^{m}|f_k'(e^{i\theta})|^2\,, \qquad (z=re^{i\psi})\,, \tag{5}$$

$$E(e^{i\theta}) \neq 0 \quad \text{almost everywhere, if one of } f_k(z) \not\equiv \text{const.}, \tag{6}$$

which we assume in the following.

If $\dfrac{dx_k(e^{i\theta})}{d\theta}$ exists, which occurs almost everywhere by the absolute continuity of $x_k(e^{i\theta})$, then by Fatou's theorem[3],

$$\lim_{z\to e^{i\theta}} \frac{\partial x_k(z)}{\partial \psi}=\frac{dx_k(e^{i\theta})}{d\theta} \qquad (z=re^{i\psi})\,,$$

when z tends to $e^{i\theta}$ non-tangentially to $|z|=1$.

Hence by (5) and (6),

$$E(e^{i\theta})=\sum_{k=1}^{m}\left(\frac{dx_k(e^{i\theta})}{d\theta}\right)^2 \neq 0 \text{ almost everywhere.} \tag{7}$$

Since $x_k(e^{i\theta})$ are absolutely continuous, $L=\int_0^{2\pi}\sqrt{E(e^{i\theta})}\,d\theta$, so that a null set on $|z|=1$ corresponds to a null set on Γ.

Next we will prove that a null set on Γ corresponds to a null set on $|z|=1$. Let e be a null set on Γ which corresponds to E on $|z|=1$ and e' be a null set which contains e and is G_δ, which corresponds to E' on $|z|=1$. Then E' contains E and being the continuous image of G_δ is G_δ and hence is measurable. Hence if we deduce $mE'=0$ from $me'=0$, then

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1) T. Radó: On Plateau's problem. Annals of Math. **31** (1930).

2) F. Riesz: Über die Randwerte einer analytischen Funktion. Math. Z. **18** (1923).

3) Fatou: Séries trigométriques et séries de Taylor Acta Math. **30** (1906).

$mE=0$ follows a fortiori, so that we assume that E is measurable. Since $me=0$, we can cover e by a sequence of open intervals Δs_n, such that $\sum_{n=1}^{\infty}|\Delta s_n|<\varepsilon$, where $|\Delta s_n|$ denotes the arc length of Δs_n. Let $\Delta\theta_n$ correspond to Δs_n on $|z|=1$, then $|\Delta s_n|=\int_{\Delta\theta_n}\sqrt{E(e^{i\theta})}\,d\theta$, so that

$$\varepsilon>\sum_{n=1}^{\infty}|\Delta s_n|=\sum_{n=1}^{\infty}\int_{\Delta\theta_n}\sqrt{E(e^{i\theta})}\,d\theta\geqq\int_E\sqrt{E(e^{i\theta})}\,d\theta\,.$$

Since ε is arbitrary, we have $\int_E\sqrt{E(e^{i\theta})}\,d\theta=0$ and from (7), it follows that $mE=0$, q. e. d.

3. Let $f_k(z)=x_k(z)+iy_k(z)$, then $f'_k(z)=\frac{\partial x_k}{\partial u}-i\frac{\partial x_k}{\partial v}$.

Since from (5) and (6), $\sum_{k=1}^{m}|f'_k(e^{i\theta})|^2\neq 0$ almost everywhere, we assume that $E(1)=\frac{1}{2}\sum_{k=1}^{m}|f'_k(1)|^2\neq 0$ and put

$$\lim_{z\to 1}\frac{\partial x_k}{\partial u}=A_k\,,\quad \lim_{z\to 1}\frac{\partial x_k}{\partial v}=B_k\,,$$

when z tends to 1 non-tangentially to $|z|=1$. Then by (1)

$$\sum_{k=1}^{m}A_k^2=\sum_{k=1}^{m}B_k^2=E(1)\neq 0\,,\quad \sum_{k=1}^{m}A_kB_k=0\,. \tag{8}$$

Let $\delta\mathfrak{z}$, $\delta\mathfrak{z}'$ be two vectors on the z-plane whose initial points are $z=1$ and end points are $z=(1-\rho\cos\theta)+i\rho\sin\theta$ and $z'=1-\rho$ respectively, then $\delta\mathfrak{z}$ makes an angle θ with $\delta\mathfrak{z}'$.

Let $\delta\mathfrak{x}=(\delta x_1,\dots,\delta x_m)$, $\delta\mathfrak{x}'=(\delta x'_1,\dots,\delta x'_m)$ correspond to $\delta\mathfrak{z}$, $\delta\mathfrak{z}'$ on S, then

$$\delta x_k=x_k(z)-x_k(1)=\frac{\partial x_k(\xi)}{\partial u}(-\rho\cos\theta)+\frac{\partial x_k(\xi)}{\partial v}\rho\sin\theta$$

$$=(-A_k\cos\theta+B_k\sin\theta)\rho+o(\rho)\,, \tag{9}$$

where ξ is a point on $\delta\mathfrak{z}$.

Similarly

$$\delta x'_k=-A_k\rho+o(\rho)\,. \tag{10}$$

Hence if we denote the angle between $\delta\mathfrak{x}$, $\delta\mathfrak{x}'$ by ϕ, then by (8), (9) and (10),

$$\lim_{\rho\to 0}\cos\phi=\lim_{\rho\to 0}\frac{\sum_{k=1}^{m}\delta x_k\delta x'_k}{\sqrt{\sum_{k=1}^{m}\delta x_k^2\sum_{k=1}^{m}\delta x'^2_k}}=\cos\theta\,,$$

or

$$\lim_{\rho\to 0}\phi=\theta\,, \tag{11}$$

and

$$\lim_{\rho\to 0}\frac{|d\mathfrak{x}|}{|d\mathfrak{z}|}=\lim_{\rho\to 0}\frac{\sqrt{\sum_{k=1}^{m}\delta x_k^2}}{\rho}=\sqrt{E(1)}\neq 0\,. \tag{12}$$

From (11), (12) we have the following theorem.

Theorem II. Under the same condition as Theorem I, the mapping of $|z|\leqq 1$ *on* S *is conformal at almost all points on* $|z|=1$.