PAPERS COMMUNICATED

56. *Note on Banach Spaces (III): A Proof of Tietze-Matsumura's Theorem.*

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A subset S of a linear metric space E is called *locally convex*, if and only if, for every $p \in S$ there exists a sphere K with center p such that $S \frown K$ is convex. In the case of the Euclidean n-spaces H. Tietze[1)] and S. Matsumura[2)] proved, that every closed, connected and locally convex set is convex.

In the present note we extend this theorem into the following form:

Theorem. If E is uniformly convex and S is compact, closed and connected set, then the local convexity of S implies the convexity in the large.

To prove the theorem we choose a finite covering by spheres $\{K_i\}$[3)] such that $S \frown K_i$ is convex; this is possible always, since the set S is compact. If a set $S \frown K_i \frown K_j$ for $i \neq j$ is non-void, then we call it a *shoal*. It is evident that a shoal is compact and closed.

On the other hand, we define a *bridge* as a continuous image of $[0, 1]$ to S, which contains only a finite number of line-segments — called *girders* —, pass through a shoal once only and joint points of girders — called *piles* — lie in shoals. For the sake of simplicity, we assume a shoal contains only one pile and even if a girder pass through a shoal, we join a pile on it.

Then obviously a bridge can be represented by an ordered set of piles and end-points such that

$$I=(p_0, p_1, \cdots, p_n).$$

Next, we define the *length of bridge* by

$$|I|=\sum_{i=1}^{n}|p_i-p_{i-1}|.$$

Since, as remarked above, all shoals are compact, we can find a bridge from a to b with minimal length. Hence to prove the theorem it is sufficient to show the following

Lemma. Every bridge with minimal length between two points of S is itself a line-segment.

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1) H. Tietze, Math. Zeits., **28** (1928), 697–707.

2) S. Matsumura (Nakajima), Tôhoku M. J., **28** (1928), 266–268.

3) We assume here K_i's sre closed spheres.

To prove this, we use the induction over the numbers of piles of minimal length' bridge. Since $n=1$ is trivial, we begin with $n=2$. Suppose the contrary is hold and $I=(a, p, b)$, then by the assumption of local convexity, we have a sphere K with center p such that $K \frown S$ is convex. We take on $\overline{ap}$ and $\overline{bp}$ two points c and d respectively such that $c \neq p \neq d$ and $c, d \in K$. Now we put $J=(a, c, d, b)$, then $|J| < |I|$ by the assumption of uniform convexity.

On the other hand, we can find a bridge J' for any J such that $|J'| \leqq |J|$, we have $|J'| < |J|$. This is a contradiction.

The remainder of the proof is almost trivial. If the lemma is proved for n, and I is a bridge of minimal length in the form $I=(p_0, p_1, \cdots, p_n, p_{n+1})$, then $I'=(p_0, \cdots, p_n)$ and $I''=(p_1, \cdots, p_{n+1})$ are bridge of minimal length between (p_0, p_n) and (p_1, p_{n+1}). Thus by the assumption of induction I' and I'' are line-segments and have a non-void subset $I' \frown I''$ in common, hence $I=I' \smile I''$ is a line-segment. This completes the proof.