70. *On the Representation of the Vector Lattice.*

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§ 1. *Introduction.* In a preceding note the author gave, jointly with M. Fukamiya[1], a representation of the vector lattice with an *Archimedean-unit* to obtain an algebraic proof of Kakutani-Krein's lattice-theoretic characterisation[2] of the space of continuous functions on a bicompact Hausdorff space. Recently, by different approaches, H. Nakano[3] and F. Maeda-T. Ogasawara[4] treated a more general case when the existence of an Archimedean-unit is not assumed. The purpose of the present note is to show that our method is also applicable to this case as a short-cut to the representation theory. Their representation space is totally disconnected and so their results will not be a direct extension of our preceding note. T. Nakayama, who stressed[5] the applicability of the Lorenzen-Clifford's procedure to the representation of the vector lattice, kindly read the manuscript and discussed with me the difference of their method and that of ours. The conclusion may, in short, be stated as follows. The point of their representation space is perhaps, so to speak, a *minimal prime ideal*, while our point is a *maximal prime ideal*. In concluding the introduction I express may hearty thanks to T. Nakayama.

§ 2. *Preliminaries.* A vector lattice E is a real linear space, some of whose elements f are *non-negative* (written $f \geqq 0$) and in which

(V 1): If $f \geqq 0$ and $\alpha \geqq 0$, then $\alpha f \geqq 0$.

(V 2): If $f \geqq 0$ and $-f \geqq 0$, then $f=0$.

(V 3): If $f \geqq 0$ and $g \geqq 0$, then $f+g \geqq 0$.

(V 4): E is a lattice by the semi-order relation $f \geqq g$ $(f-g \geqq 0)$.

We put, as usual, $|f|=f^+-f^-$, $f^+=f \vee 0$, $f^-=f \wedge 0$. Two elements f and g is called *disjoint* (or *orthogonal*) if $|f| \wedge |g|=0$. Let $\{u_a\}$ be a maximal set of mutually disjoint positive ($u_a \geqq 0$ but $\neq 0$) elements of E. The maximality means that if $x>0$ then $x \wedge u_a > 0$ for at least one u_a. An element f is called *nilpotent* (with respect to $\{u_a\}$) if $n(|f| \wedge u_a) < u_a$ $(n=1, 2, \dots)$ for all u_a. The totality R of the nilpotent elements is called the *radical* of E. R constitutes a linear subspace of E. *Proof*: Let f and g be nilpotent, then $n(|f+g| \wedge u_a) \leqq n\big(2(|f| \vee |g|) \wedge u_a\big) \leqq 2n\big((|f| \wedge u_a) \vee (|g| \wedge u_a)\big) < u_a \vee u_a = u_a$ $(n=1,$

1) Proc. **17** (1941), 479-482.

2) S. Kakutani: Proc. **16** (1940), 63-67. M. and S. Krein: C. R. URSS, **27** (1940), 427-430.

3) Proc. Physico-Math. Soc. Japan, **23** (1941), 485-511.

4) 全國紙上數學談話會, 第 231 號.

5) 全國紙上數學談話會, 第 233 號.

2, ...). Moreover R is an *ideal* of E, viz. $f \in R$ and $|g| \leqq |f|$ implies $g \in R$.

Lemma 1. Let N be a linear subspace of E. Then the *linear-congruence* $a \equiv b$ (mod N) is also a *lattice-congruence*:

$$c \equiv c', \quad d \equiv d' \pmod N \quad \text{implies } c \begin{smallmatrix}\vee\\\wedge\end{smallmatrix} d \equiv c' \begin{smallmatrix}\vee\\\wedge\end{smallmatrix} d' \pmod N$$

if and only if N is an ideal of E.

Proof. See, for example, Garrett Birkhoff: Lattice Theory (1940), 109.

An ideal $N \neq E$ is called *prime*, if the *residual vector lattice* E/N of E mod N is *simply ordered*, viz. $f \geqq g$ or $f < g$ (mod N) for any two elements f, g. Since $x^+ \wedge (-x)^+ = 0$ for any x, we see that E is simply ordered if and only if $|f| \wedge |g| = 0$ implies $f=0$ or $g=0$.

Lemma 2[1]. For any $f \neq 0$, there exists a prime ideal $N \bar{\ni} f$.

Proof. Let E be not simply ordered and suppose $g > 0$, $h > 0$, $g \wedge h = 0$. Then at least one of the ideals

$$N^{(1)} = \mathop{\mathfrak{E}}_{g'}(|g'| \leqq \alpha g, \alpha < \infty) \neq 0, \quad N^{(2)} = \mathop{\mathfrak{E}}_{h'}(|h'| \leqq \alpha h, \alpha < \infty) \neq 0$$

does not contain f. Let $N_1 = N^{(1)} \bar{\ni} f$ and let $0 \subset N_1 \subset N_2 \subset \cdots \subset N_\eta \subset \cdots$ $(\eta < \omega)$ be a properly increasing (transfinite) sequence of ideals not containing f. If ω is a limit ordinal, define $x \equiv y$ (mod N_ω) to mean $x \equiv y$ (mod N_η) for some $\eta < \omega$. N_ω does not contain f. Thus we may obtain, at a certain step, an ideal $N \bar{\ni} f$ which is not contained in no other ideal $\bar{\ni} f$. By this maximality N is a prime ideal.

§ 3. *The representation theorem.* By the lemma 2, there exists an ideal $N \bar{\ni} u_\alpha$ which is not contained in no other ideal $\bar{\ni} u_\alpha$. Let $\mathfrak{N}(u_\alpha)$ be the totality of such ideals and let $\mathfrak{N}$ be the totality of the ideals $\in$ some $\mathfrak{N}(u_\alpha)$. Since each $N \in \mathfrak{N}$ is a prime ideal, there exists, for any $N \in \mathfrak{N}$, exactly one $u_\alpha = u_{\alpha(N)}$ which satisfies $u_{\alpha(N)} \bar{\in} N$. Thus we may write u_N for $u_{\alpha(N)}$. For any $x \in E$ and for any $N \in \mathfrak{N}$ we put

$$(1) \quad \begin{cases} x(N) = \text{l. u. b. } \lambda, \text{ where } x \geqq \lambda u_N \pmod N \\ \qquad\;\; = \text{g. l. b. } \mu, \text{ where } x \leqq \mu u_N \pmod N. \end{cases}$$

The equivalence of the two definitions of $x(N)$ follows from the fact that E/N is simply ordered. Of course, we put $x(N) = +\infty$ if there exists no μ such that $x \leqq \mu u_N$ (mod N); similarly for $x(N) = -\infty$. By the lemma 1, we have

$$(2) \quad (x \vee y)(N) = \max\big(x(N), y(N)\big), \quad (x \wedge y)(N) = \min\big(x(N), y(N)\big),$$

$$(3) \qquad (\alpha x + \beta y)(N) = \alpha x(N) + \beta y(N).$$

It is to be noted that (3) is ambiguous in case $x(N) = \pm\infty$, $y(N) = \pm\infty$. If E satisfies the *Archimedean axiom* (V 6) below, this ambiguity will be removed by introducing a topology in $\mathfrak{N}$ (§ 4).

Remark. Let there exist an *Archimedean-unit* u:

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1) 全國紙上數學談話會, 第 227 號.

(V 5): $\begin{cases} \text{For any } x \in E, \text{ there exists a positive number } \alpha = \alpha(x) \text{ such} \\ \text{that } -\alpha(x)u \leqq x \leqq \alpha(x)u. \end{cases}$

If we take the one-element-set $\{u\}$ for $\{u_a\}$, then every function $x(N)$ is bounded on $\mathfrak{N}\big(|x(N)| \leqq \alpha(x)\big)$. In this (Archimedean-unit) case, $N \in \mathfrak{N}$ means that N is a *maximal non-trivial* ideal.

Returning to our representation (1), we have

(4) $x(N) = 0$ identically on $\mathfrak{N}$ if and only if $x \in R$.

Proof. Let $x \geqq 0$ be nilpotent, then, by (1) and (2), we have $n\Big(\min\big(x(N), 1\big)\Big) \leqq 1$ $(n = 1, 2, \dots)$ and hence $x(N) = 0$ on $\mathfrak{N}$. Conversely let $0 \leqq x \leqq u_a$ and $nx \not\leqq u_a$, $n \geqq 1$. By the lemma 2, there exists a prime ideal $N(y) \bar{\ni} y = (nx - u_a)^+$, viz. $(nx - u_a) > 0$ $\big(\mathrm{mod}\ N(y)\big)$. $N(y)$ does not contain u_a, for otherwise, we would obtain $0 = (0-0) > 0$ $\big(\mathrm{mod}\ N(y)\big)$. Let N be an ideal $\geqq N(y)$, $\bar{\ni} u_a$, which is not contained in no other ideal $\geqq N(y)$, $\bar{\ni} u_a$. Surely we have $N \in \mathfrak{N}$ and hence $u_a = u_N$. Since $N \geqq N(y)$, we have $(nx - u_N) \geqq 0$ $(\mathrm{mod}\ N)$ and thus $nx(N) \geqq u_N(N)$ or $x(N) \geqq 1/n$.

§ 4. *Introduction of a topology and the Archimeden axiom.* For any $x \geqq 0$, we call x-set the totality of $N \in \mathfrak{N}$ such that $N \bar{\ni} x$, Then we have

(5) $\begin{cases} (x \vee y)\text{-set} = \text{the sum } (x\text{-set}) \vee (y\text{-set}), \\ (x \wedge y)\text{-set} = \text{the intersection } (x\text{-set}) \wedge (y\text{-set}). \end{cases}$

Proof. That $(x\text{-set}) \vee (y\text{-set}) \leqq (x \vee y)\text{-set}$ is evident from the definition of the ideal. Let $x \vee y \bar{\in} N$ and let $x \in N$, $y \in N$. Then, since N is prime, $x \vee y \equiv x$ or y $(\mathrm{mod}\ N)$, that is, $x \vee y \in N$, contrary to the hypothesis. Next we have $(x\text{-set}) \wedge (y\text{-set}) \geqq (x \wedge y)\text{-set}$ from the definition of the ideal. Let $x \bar{\in} N$, $y \bar{\in} N$, then, since N is prime, $x \wedge y \equiv x$ or y $(\mathrm{mod}\ N)$ and thus $x \wedge y \bar{\in} N$. Q. E. D.

Hence, if we call *open* the x-set's, $\mathfrak{N}$ is a topological space. In the truth, $\mathfrak{N}$ is a *Hausdorff space*. *Proof*: If $N_1 \neq N_2$, then there exist $x_1 > 0$ and $x_2 > 0$ such that $x_1 \bar{\in} N_1$, $x_1 \in N_2$, $x_2 \bar{\in} N_2$, $x_2 \in N_1$. Since N_1 is prime, $(x_1 - x_2) \geqq 0$ $(\mathrm{mod}\ N_1)$ or $(x_1 - x_2) < 0$ $(\mathrm{mod}\ N_1)$. The latter inequality is excluded by $x_1 > 0$ $(\mathrm{mod}\ N_1)$, $x_2 = 0$ $(\mathrm{mod}\ N_1)$. Thus $(x_1 - x_2)^+ \bar{\in} N_1$ and $(x_2 - x_1)^+ \bar{\in} N_2$ similarly. By the idantity $x^+ \wedge (-x)^+ = 0$ and (5), the intersection of $(x_1 - x_2)^+$-set and $(x_2 - x_1)^+$-set is void.

The continuity of the function $x(N)$ on $\mathfrak{N}$ may be proved as follows. Let $x(N_0) = \lambda \neq \pm\infty$ and let ε be any positive number. Then we have $(\lambda - \varepsilon) \leqq x(N) \leqq (\lambda + \varepsilon)$ if N belongs to

$$u_{N_0} \wedge \Big(\big(x - (\lambda - \varepsilon)u_{N_0}\big)^+\Big) \wedge \big((\lambda + \varepsilon)u_{N_0} - x)^+\big)\text{-set} \ni N_0 .$$

Similarly for the case $x(N_0) = \pm\infty$.

Next we assume that the *Archimedean axiom*:

(V 6): $\bigwedge_{n\geqq 1}\left(\frac{1}{n}x\right)=0$ for any $x\geqq 0$

is satisfied in E. We have, in this case, $R=0$. Moreover we have the result:

(6) The set of N at which $x(N)=\pm\infty$ is non-dense on $\mathfrak{R}$.

Proof. We assume $x>0$ and will prove that, for any $y>0$, there exists a point $N_0\in y$-set such that $x(N_0)<+\infty$. Assume the contrary and let $x>nu_N$ (mod N) ($n=1, 2, \dots$) for every $N\in y$-set. Then

(*) $x>n(u_N\wedge y)$ (mod N) ($n=1, 2, \dots$) for every $N\in y$-set.

By the maximality of $\{u_a\}$, there exists u_a such that $u_a\wedge y>0$. By (V 6), we have $x\ngeqq z=n(u_a\wedge y)$ for some $n\geqq 1$. Thus $(z-x)^+>0$ and hence, by $R=0$, $\max\left\{\left(z(N_0)-x(N_0)\right),\ 0\right\}>0$ for some N_0. Therefore

$$n\cdot\min\left(u_a(N_0), y(N_0)\right)>x(N_0)\geqq 0\,.$$

This contradicts to (*), for from $u_a(N_0)>0, y(N_0)>0$ we must have $u_a=u_{N_0}$, $N_0\in y$-set.

Remark 1. In general, our vicinity, the x-set, does not disconnect the space $\mathfrak{R}$. While, in the treatments of H. Nakano and F. Maeda-T. Ogasawara cited above, the representation space is totally disconnected.

Remark 2. In the Archimedean-unit case, our topology is equivalent to the *weak topology* obtained by calling open the set of the form

$$\underset{N}{\mathfrak{E}}\left(|x_i(N)-x_i(N_0)|<\varepsilon_i \quad (i=1, 2, \dots, n)\right)$$

where $-u\leqq x_i\leqq u$, $\varepsilon_i>0$ ($i=1, 2, \dots, n$) and n are arbitrary. In this case, $\mathfrak{R}$ is bicompact and any continuous function on $\mathfrak{R}$ may be approximated uniformly on $\mathfrak{R}$ by the functions $x(N)$, $x\in E$. For the proof, see the preceding note.