89. Concircular Geometry V. Einstein Spaces.

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§0. In four recent papers[1], we have defined the concircular transformations in the Riemannian spaces and studied the so-called concircular geometry in these spaces.

The concircular transformation is defined as a conformal transformation

(0.1) $$\bar{g}_{\mu\nu}=\rho^2 g_{\mu\nu}$$

of Riemannian metric which satisfies the relation

(0.2) $$\rho_{\mu\nu}=\phi g_{\mu\nu}\,,$$

where

(0.3) $$\rho_{\mu\nu}=\rho_{\mu;\,\nu}-\rho_\mu\rho_\nu+\frac{1}{2}g^{\alpha\beta}\rho_\alpha\rho_\beta g_{\mu\nu}\,,\qquad (\rho_\mu=\partial\log\rho/\partial u^\mu)$$

and the semi-colon means the covariant derivative with respect to the Christoffel symbols $\{^{\,\lambda}_{\mu\nu}\}$ formed with $g_{\mu\nu}$, ϕ being a certain function of the coordinates u^λ.

The concircular transformation (0.1) of the metric keeps unchanged the differential equations

(0.4) $$\frac{\delta^3u^\lambda}{\delta s^3}+\frac{du^\lambda}{ds}\,g_{\mu\nu}\,\frac{\delta^2u^\mu}{\delta s^2}\,\frac{\delta^2u^\nu}{\delta s^2}=0$$

of the geodesic circle where $\delta/\delta s$ denotes the covariant differentiation along the curve, s being the arc length of the curve.

If a Riemannian space V_n admits the concircular transformation (0.1), it will be readily proved that the tensor

(0.5) $$Z^\lambda_{\mu\nu\omega}=R^\lambda_{\mu\nu\omega}-\frac{R}{n(n-1)}(g_{\mu\nu}\delta^\lambda_\omega-g_{\mu\omega}\delta^\lambda_\nu)\,,$$

and consequently the contracted tensor

(0.6) $$Z_{\mu\nu}=Z^\lambda_{\mu\nu\lambda}=R_{\mu\nu}-\frac{R}{n}\,g_{\mu\nu}$$

are invariant under these concircular transformations, where

(0.7) $$R^\lambda_{\mu\nu\omega}=\{^{\,\lambda}_{\mu\nu}\}_{,\,\omega}-\{^{\,\lambda}_{\mu\omega}\}_{,\,\nu}+\{^{\,\alpha}_{\mu\nu}\}\{^{\,\lambda}_{\alpha\omega}\}-\{^{\,\alpha}_{\mu\omega}\}\{^{\,\lambda}_{\alpha\nu}\}$$

and

(0.8) $$R_{\mu\nu}=R^\lambda_{\mu\nu\lambda}\,,\qquad R=g^{\mu\nu}R_{\mu\nu}\,,$$

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1) K. Yano: Concircular geometry I. Concircular transformations, Proc. **16** (1940), 195–200, II. Integrability conditions of $\rho_{\mu\nu}=\phi g_{\mu\nu}$, ibid. pp. 354–360, III. Theory of curves, ibid. pp. 442–445, IV. Theory of subspaces, ibid. pp. 505–511. These papers will be cited as C. G. I, II, III and IV.

the comma representing the ordinary partial differentiation.

The vanishing of the tensor $Z^{\lambda}_{\mu\nu\omega}$ is the necessary and sufficient condition that the Riemannian space V_n may be transformed into a Euclidean space by a suitable concircular transformation. In the other words, the necessary and sufficient condition that the space be concircularly flat is that the space be of constant Riemannian curvature.

Furthermore the invariance of the tensors $Z^{\lambda}_{\mu\nu\omega}$ and $Z_{\mu\nu}$ shows that the space of constant Riemannian curvature and Einstein space are transformed into the space of constant Riemannian curvature and Einstein space respectively by any concircular transformations.

In C. G. II, we have shown that the necessary and sufficient condition whereon a Riemannian space admits a solution of the partial differential equations (0.2) is that the Riemannian space contain a family of totally umbilical hypersurfaces whose orthogonal trajectories are geodesic Ricci-curves.

If these conditions are satisfied, the fundamental quadratic differential form of V_n may be reduced to

$$ds^2 = g_{\mu\nu}(u^\lambda)du^\mu du^\nu = \sigma(u^n)g^*_{jk}(u^i)du^j du^k + g_{nn}(u^n)du^n du^n\,, \tag{0.9}$$

where the greek indices run from 1 to n while latin ones from 1 to $n-1$. This convention will be followed throuthout the paper.

The quadratic differential form (0.9) was also obtained by A. Fialkow.[1]

§1. On the other hand, H. W. Brinkmann[2] has studied the Einstein spaces which are mapped conformally on each other. But, his results are quite analytical and we shall again take up this problem and will study, in the scheme of the concircular geometry, the geometrical properties of the Einstein spaces which are conformal to another Einstein spaces.

Consider a general Riemannian space V_n whose fundamental quadratic differential form is given by

$$ds^2 = g_{\mu\nu}du^\mu du^\nu\,. \tag{1.1}$$

When we effect a conformal transformation

$$\bar{g}_{\mu\nu} = \rho^2 g_{\mu\nu} \tag{1.2}$$

of the metric tensor $g_{\mu\nu}$, then the curvature tensor $R^{\lambda}_{\mu\nu\omega}$ will be transformed into $\bar{R}^{\lambda}_{\mu\nu\omega}$ according to

$$\bar{R}^{\lambda}_{\mu\nu\omega} = R^{\lambda}_{\mu\nu\omega} - \rho_{\mu\nu}\delta^{\lambda}_{\omega} + \rho_{\mu\omega}\delta^{\lambda}_{\nu} - g_{\mu\nu}\rho^{\lambda}_{\cdot\omega} + g_{\mu\omega}\rho^{\lambda}_{\cdot\nu}\,, \tag{1.3}$$

where

$$\rho_{\mu\nu} = \rho_{\mu;\,\nu} - \rho_\mu\rho_\nu + \frac{1}{2}g^{\alpha\beta}\rho_\alpha\rho_\beta g_{\mu\nu}\,, \qquad \rho^{\lambda}_{\cdot\omega} = g^{\lambda\nu}\rho_{\nu\omega}\,, \tag{1.4}$$

and

$$\rho_\nu = \partial \log \rho / \partial u^\nu\,. \tag{1.5}$$

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1) A. Fialkow: Conformal geodesics, Trans. Amer. Math. Soc. **45** (1939) pp. 443–473.

2) H. W. Brinkmann: Einstein spaces which are mapped conformally on each other, Math. Ann. **94** (1925) pp. 119–145.

From (1.3), we have

$$(1.6)\qquad -\frac{\bar{R}_{\mu\nu}}{n-2}+\frac{\bar{R}\bar{g}_{\mu\nu}}{2(n-1)(n-2)}=-\frac{R_{\mu\nu}}{n-2}+\frac{Rg_{\mu\nu}}{2(n-1)(n-2)}+\rho_{\mu\nu},$$

where

$$(1.7)\qquad \bar{R}_{\mu\nu}=\bar{R}^{\lambda}_{\mu\nu\lambda} \quad \text{and} \quad \bar{R}=\bar{g}^{\mu\nu}\bar{R}_{\mu\nu}.$$

Thus, putting

$$(1.8)\qquad \Pi^{0}_{\mu\nu}=-\frac{R_{\mu\nu}}{n-2}+\frac{Rg_{\mu\nu}}{2(n-1)(n-2)},$$

we have

$$(1.9)\qquad \bar{\Pi}^{0}_{\mu\nu}=\Pi^{0}_{\mu\nu}+\rho_{\mu\nu}.$$

If, in particular, the Riemannian spaces $\bar{V}_n$ and V_n are both Einstein spaces, we have

$$(1.10)\qquad \bar{R}_{\mu\nu}=\frac{1}{n}\bar{R}\bar{g}_{\mu\nu} \quad \text{and} \quad R_{\mu\nu}=\frac{1}{n}Rg_{\mu\nu}.$$

In this case, the tensor $\Pi^{0}_{\mu\nu}$ must be of the form

$$(1.11)\qquad \Pi^{0}_{\mu\nu}=-\frac{1}{2}Kg_{\mu\nu},$$

where

$$(1.12)\qquad K=\frac{1}{n(n-1)}R$$

is a constant.

Consequently, the equation (1.9) shows that

$$(1.13)\qquad \rho_{\mu\nu}=-\frac{1}{2}(\bar{K}\rho^{2}-K)g_{\mu\nu}.$$

Thus, if an Einstein space is conformal to an another Einstein space, this Einstein space must admit a concircular transformation.

We shall then study the Einstein spaces admitting the concircular transformations. The results are quite similar to those of Brinkmann, but it seems to be interesting that our results may have some relations with the spaces which Einstein and Bergmann have considered in their new unified field theory.[1)]

§2. The space being Einsteinian, we have, from the theorem IV in C. G. II,

Theorem. The necessary and sufficient condition whereon an Einstein space admits the concircular transformations is that the space contain a family of ∞^1 *totally-umbilical hypersurfaces whose orthogonal trajectories are geodesics.*

When this condition is satisfied, the fundamental quadratic differential form must be of the form

$$(2.1)\qquad ds^{2}=\sigma(u^{n})g^{*}_{jk}(u^{i})du^{j}du^{k}+g_{nn}(u^{n})du^{n}du^{n}.$$

1) A. Einstein and P. Bergmann: On a generalization of Kaluza's theory of electricity. Annals of Math. **39** (1938) pp. 683-701.

If we suppose that the quadratic differential form be positive definite, we can put (2.1) in the form

$$(2.2)\qquad ds^2=f^2(u^n)g^*_{jk}du^j du^k+du^n du^n$$

by effecting a transformation of the form

$$u^n \rightarrow \int \sqrt{g_{nn}}\, du^n$$

The fundamental quadratic differential form being (2.2), we can calculate all the Christoffel symbols $\{^{\lambda}_{\mu\nu}\}$ as follows:

$$(2.3)\qquad \begin{cases} \{^{i}_{jk}\}=\{^{i}_{jk}\}^*, \quad \{^{n}_{jk}\}=-ff'g^*_{jk}, \quad \{^{i}_{jn}\}=\{^{i}_{nj}\}=\dfrac{f'}{f}\delta^i_j, \\ \{^{i}_{nn}\}=\{^{n}_{jn}\}=\{^{n}_{nj}\}=\{^{n}_{nn}\}=0, \end{cases}$$

where $\{^{i}_{jk}\}^*$ are the Christoffel symbols formed with g^*_{jk} and dashes denote the differentiation with respect to u^n.

Thus, the components $R^{\lambda}_{\mu\nu\omega}$ of the Riemann-Christoffel curvature tensor are give by

$$\begin{cases} R^i_{jkh}=R^{*i}_{jkh}-f'^2(g^*_{jk}\delta^i_h-g^*_{jh}\delta^i_k), \quad R^n_{jkn}=-ff''g^*_{jk} \\ R^i_{jnk}=0, \quad R^n_{jnn}=0, \\ R^i_{nnh}=-\dfrac{f''}{f}\delta^i_h, \quad R^n_{nnn}=0, \end{cases}$$

from which we find

$$(2.4)\qquad \begin{cases} R_{jk}=R^*_{jk}-[(n-2)f'^2+ff'']g^*_{jk}, \\ R_{jn}=0, \\ R_{nn}=-(n-1)\dfrac{f''}{f}, \end{cases}$$

where R^{*i}_{jkh} and R^*_{jk} mean the Riemann-Christoffel curvature tensor and Ricci tensor respectively formed with $\{^{i}_{jk}\}^*$

We have, on the other hand,

$$R_{\mu\nu}=\frac{1}{n}Rg_{\mu\nu},$$

consequently, the first two equations of (2.4) may be written as

$$(2.5)\qquad \begin{cases} R^*_{jk}=[(n-2)f'^2+ff''+(n-1)Kf^2]g^*_{jk}, \\ R^*_{jn}=0, \end{cases}$$

and the last one as

$$(2.6)\qquad f''=-Kf.$$

Substituting (2.6) into the first equation of (2.5), we have

$$(2.7)\qquad R^*_{jk}=(n-2)(f'^2+Kf^2)g^*_{jk}.$$

Here, we must distinguish the three cases $K>0$, $K=0$ and $K<0$.

Case I. $K=\dfrac{1}{n(n-1)}R>0.$

Then the equation (2.6) gives

$$f=A\cos\sqrt{K}\,u^n+B\sin\sqrt{K}\,u^n, \tag{2.8}$$

and consequently (2.7) becomes

$$R^*_{jk}=(n-2)(A^2+B^2)Kg^*_{jk}, \tag{2.9}$$

from which we have

$$K^*=\frac{1}{(n-1)(n-2)}R^*=(A^2+B^2)K. \tag{2.10}$$

Thus we have the

Theorem. The necessary and sufficient cindition whereon an Einstein space whose scalar curvature is positive admits a concircular transformation is that the fundamental quadratic differential form may be reduced to

$$ds^2=(A\cos\sqrt{K}\,u^n+B\sin\sqrt{K}\,u^n)^2g^*_{jk}(u^i)du^jdu^k+du^ndu^n, \tag{2.11}$$

the Riemannian space V^*_{n-1} whose fundamental quadratic differential form is $g^*_{jk}(u^i)du^jdu^k$ being also an Einstein space with positive scalar curvature $R^*=\frac{n-2}{n}(A^2+B^2)R$.

Case II. $K=\frac{1}{n(n-1)}R=0.$

In this case, the equation (2.6) gives

$$f''=0,$$

from which we have

$$f=Au^n+B. \tag{2.12}$$

Substituting this equation in (2.7), we find

$$R^*_{jk}=(n-2)A^2g^*_{jk}, \tag{2.13}$$

from which we have

$$K^*=\frac{1}{(n-1)(n-2)}R^*=A^2 \tag{2.14}$$

Thus we have the

Theorem. The necessary and sufficient condition whereon an Einstein space whose scalar curvature is zero admits a concircular transformation is that the fundamental quadratic differential form may be reduced to

$$ds^2=(Au^n+B)^2g^*_{jk}(u^i)du^jdu^k+du^ndu^n, \tag{2.15}$$

the Riemannian space V^*_{n-1} whose fundamental quadratic differential form is $g^*_{jk}(u^i)du^jdu^k$ being also an Einstein space with positive scalar curvature $R^*=(n-1)(n-2)A^2$.

Case III. $K=\frac{1}{n(n-1)}R<0.$

In this case, the equation (2.6) gives

(2.16) $$f = Ae^{\sqrt{-K}u^n} + Be^{-\sqrt{-K}u^n}.$$

Substituting this equation in (2.5), we have

(2.17) $$R^*_{jk} = 4(n-2)ABKg^*_{jk},$$

from which we have

(2.18) $$K^* = \frac{1}{(n-1)(n-2)} R^* = 4ABK.$$

Thus, we have the

Theorem. The necessary and sufficient condition whereon an Einstein space whose scalar curvature is negative admits a concircular transformation is that the fundamental quadratic differential form may be reduced to

(2.19) $$ds^2 = (Ae^{\sqrt{-K}u^n} + Be^{-\sqrt{-K}u^n})^2 g^*_{jk}(u^i)du^j du^k + du^n du^n,$$

the Riemannian space V^*_{n-1} whose fundamental quadratic differential form is $g^*_{jk}(u^i)du^j du^k$ being also an Einstein space with scalar curvature $R^* = \frac{4(n-2)}{n} ABR.$