105. *An Abstract Integral (VIII).*

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Introduction. In this paper we intend to establish the theory of Lebesgue integral of the vector lattice valued functions. This subject has been discussed by Bochner[1] and Izumi[2]. Our consideration differs from them in that it is based on the notion of semi-ordering.

We define the Lebesgue integral which is analogous to Young[3], Daniell[4], and Banach's[5] one in real valued functions. It is noteworthy that the integrable functions are not always approximated by step functions or Riemann integrable functions, although the integral is obtained by an extension from step functions or Riemann integrable functions. This integral includes obviously the Bochner's and, if we neglect conditions on the vector lattice, includes the Izumi's.

And moreover our considerations can be abstracted in that way which regards the extension of an integral as the extension of a linear operation between two given vector lattices. This problem has been treated by Izumi and Nakamura[6] in the case of a linear functional.

1. *The class* T_0. Let $f(t)$ be an abstract function defined in abstract space and with range in a complete regular vector lattice L[7].

We assume that the initial class T_0 of functions is closed with respect to the operations:—cf, f_1+f_2, $f_1 \cup f_2$, $f_1 \cap f_2$, and that the functions of T_0 are bounded. Further let a functional operation $I(f)$ be defined on T_0 such that

(*A*) $I(f_1+f_2)=I(f_1)+I(f_2)$;

(*L*) If $f_1 \geqq f_2 \geqq \cdots$ and $\lim f_n=0$, then $\lim I(f_n)=0$.

From these we can easily conclude that

(*C*) $I(cf)=cI(f)$, where c is a real constant;

(*P*) If $f \geqq 0$, $I(f) \geqq 0$.

Then the class T_0 is obviously a lattice. For some instances of the class T_0, we may consider the class of step functions or Riemann integrable functions.

2. *Extension to class* T_1 *from* T_0. If $f_1 \leqq f_2 \leqq \cdots$ where $f_i \in T_0$, then $\lim f_n$ exists (if we adjoin $+\infty$ to the range), and we define T_1 as class of such limit functions. For such (f_n) we have $I(f_1) \leqq I(f_2) \leqq \cdots$ and then $\lim I(f_n)$ exists (if allow $+\infty$ as limit).

1) Bochner, Nat. Acad. Sci., **26** (1940), p. 29.
2) Izumi, Proc. **18** (1942), 53.
3) Young, Proc. London Math. Soc., **18** (1914), p. 109.
4) Daniell, Ann. of Math., **19** (1917), p. 279.
5) Saks, *Theory of the Integral*, 1937, p. 320.
6) Izumi and Nakamura, Proc. **16** (1940), 518.
7) For these definitions and discussions, see Kantorovitch, Recueil Math. of Moskau, **49** (1940), p. 209, and Orihara, This Proc. Regurality is used in (4.6) and (4.7) only.

(2.1) *If* $f_1 \leqq f_2 \leqq \cdots$ $(f_i \in T_0)$ *and* $\lim f_n \geqq h \in T_0$, *then* $\lim I(f_n) \geqq I(h)$.

(2.2) *If* $f_1 \leqq f_2 \leqq \cdots$, $g_1 \leqq g_2 \leqq \cdots$, $(f_i, g_i \in T_0)$, *and* $\lim f_n \geqq \lim g_n$, *then* $\lim I(f_n) \geqq \lim I(g_n)$.

(2.3) *If* $f_1 \leqq f_2 \leqq \cdots$, $g_1 \leqq g_2 \leqq \cdots$, $(f_i, g_i \in T_0)$, *and* $\lim f_n = \lim g_n$, *then* $\lim I(f_n) = \lim I(g_n)$.

We define $I(f) = \lim I(f_n)$, if $T_1 \ni f = \lim f_n$, $f_n \in T_0$. Then evidently (P), (A) and (C) (when $c \geqq 0$) are satisfied.

(2.4) If $f_1 \leqq f_2 \leqq \cdots$, $f_i \in T_1$ and $\lim f_n = f$, then $f \in T_1$ and $I(f) = \lim I(f_n)$.

3. *Semi-integrals.* For any function f we define

$$\bar{I}(f) = \bigwedge I(\varphi),$$

where $\bigwedge$ is taken for all functions $\varphi \in T_1$, such as $\varphi \geqq f$. This is called the upper semi-integral of f. Then we have

(3.1) *If* $c > 0$, $\bar{I}(cf) = c\bar{I}(f)$.

(3.2) $\bar{I}(f_1 + f_2) \leqq \bar{I}(f_1) + \bar{I}(f_2)$.

(3.3) *If* $f \leqq g$, $\bar{I}(f) \leqq \bar{I}(g)$.

We define $\underline{I}(f) = -\bar{I}(f)$. This is called the lower semi-integral of f. Similarly we get

(3.4) $\bar{I}(f) \geqq \underline{I}(f)$.

(3.5) $\bar{I}(f \cup g) + \bar{I}(f \cap g) \leqq \bar{I}(f) + \bar{I}(g)$.

(3.6) $\bar{I}(|f|) - \underline{I}(|f|) \leqq \bar{I}(f) - \underline{I}(f)$.

4. *Integrability.* If $\bar{I}(f) = \underline{I}(f) =$ finite, f is said to be integrable and we define

$$I(f) = \bar{I}(f) = \underline{I}(f),$$

which is called the integral of f. Then we can prove the following theorems.

(4.1) *If* $f \geqq 0$ *is integrable,* $I(f) \geqq 0$.

(4.2) *If* c *is a constant and* f *is integrable, then* cf *is integrable.*

(4.3) *If* f_1, f_2 *are integrable, then* $f_1 + f_2$ *is so and* $I(f_1 + f_2) = I(f_1) + I(f_2)$.

(4.4) *If* f *is integrable, so is* $|f|$ *and* $|I(f)| \leqq I(|f|)$.

(4.5) *If* f_1, f_2 *are integrable, so are* $f_1 \cup f_2$, $f_1 \cap f_2$.

(4.6) *If* $I \leqq f_2 \leqq \cdots$ *is a sequence of integrable functions, and if* $\lim I(f_n)$ *is finite, then* $\lim f_n = f$ *is integrable and* $I(f) = \lim I(f_n)$.

While if $\lim I(f_n)=+\infty$, *then* $I(f)=+\infty$.

Proof. By $-f\leqq -f_n$ we have $\bar{I}(-f)\geqq I(-f_n)$. And then

$$\underline{I}(f)\geqq I(f_n) \quad \text{for} \quad n=1,2,\dots .$$

Hence
$$\underline{I}(f)\geqq \lim I(f_n). \tag{1}$$

This proves the last part of the theorem.

If $\varphi_a^{(1)}\geqq f_1$, and $\varphi_a^{(1)}\in T_1$, then $\bar{I}(f_1)=\bigwedge_a I(\varphi_a^{(1)})$. But since L is regular, there exists enumerable $\varphi_{a_n}^{(1)}$ such that

$$\bigwedge_a I(\varphi_a^{(1)})=\bigwedge_n I(\varphi_{a_n}^{(1)}).$$

If we put
$$\varphi_{a_1}^{(1)}=g_1^{(1)}, \qquad \varphi_{a_1}^{(1)}\frown\varphi_{a_2}^{(1)}=g_2^{(1)},\dots$$

then
$$g_1^{(1)}\geqq g_2^{(1)}\geqq\cdots, \quad \text{and} \quad g_1^{(1)}\in T_1.$$

$$\bigwedge_n I(\varphi_{a_n}^{(1)})\geqq \bigwedge_n I(g_n^{(1)})\geqq I(f_1).$$

But since
$$\bigwedge_n I(g_n^{(1)})=\lim I(g_n^{(1)}),$$

we get
$$\lim I(g_n^{(1)})=I(f_1), \quad \text{and} \quad g_n^{(1)}\geqq f_1.$$

Thus for any $e>0$ and all $n_1\geqq N_1$, there exist a U and N_1, such that

$$I(g_{n_1}^{(1)})\leqq I(f_1)+\frac{1}{2}eU, \quad \text{for} \quad n_1\geqq N_1;\quad g_{n_1}^{(1)}\geqq f_1.$$

Similarly, taking the sequence $f_2-f_1, f_3-f_2,\dots$, we have

$$I(g_{n_2}^{(2)})\leqq I(f_2-f_1)+\frac{1}{2^2}eU, \quad \text{for} \quad n_2\geqq N_2;\quad g_{n_2}^{(2)}\geqq f_2-f_1\geqq 0,$$

$$I(g_{n_3}^{(3)})\leqq I(f_3-f_2)+\frac{1}{2^3}eU, \quad \text{for} \quad n_3\geqq N_3;\quad g_{n_3}^{(3)}\geqq f_3-f_2\geqq 0,$$

....

If we put
$$\psi_m=g_{n_1}^{(1)}+g_{n_2}^{(2)}+\cdots+g_{n_m}^{(m)},$$

then
$$\psi_1\leqq\psi_2\leqq\cdots.$$

But since $\psi_m\geqq f_m$, and $\lim\psi_m\geqq\lim f_m=f$,

we have
$$\begin{aligned}I(\psi_m)&=I(g_{n_1}^{(1)})+I(g_{n_2}^{(2)})+\cdots+I(g_{n_m}^{(m)})\\&\leqq I(f_1)+I(f_2-f_1)+\cdots+I(f_m-f_{m-1})\\&\qquad+eU\Big(\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\cdots+\frac{1}{2^m}\Big)\\&\leqq I(f_m)+eU.\end{aligned}$$

Therefore
$$\lim I(\psi_m)\leqq\lim I(f_n)+eU.$$

That is
$$\bar{I}(f)\leqq\lim I(f_n)+eU.$$

$$\bar{I}(f)\leqq\lim I(f_n). \tag{2}$$

By (1) and (2), we get the integrability of f and

$$I(f)=\lim I(f_n).$$

(4.7) *If* $f_1, f_2, \cdots$ *is a sequence of integrable functions with limit* f, *and if there exists an integrable function* φ *such that* $|f_n| \leqq \varphi$ *for all* n, *then* f *is integrable and* $\lim I(f_n)=I(f)$.

Proof. If we put

$$g_{r,s}=f_r \cup f_{r+1} \cup \cdots \cup f_{r+s},$$

then
$$g_{r,s} \leqq g_{r,s+1} \leqq \cdots \rightarrow g_r$$

and
$$g_r \geqq g_{r+1} \geqq \cdots \rightarrow f.$$

But since $g_{r,s}$ is integrable and $I(g_{r,s}) \leqq I(\varphi)$, g_r is integrable. And since $-g_r \leqq -g_{r+1} \leqq \cdots$ and $-f$ is also integrable, we have $I(-f) = \lim I(-g_r)$.

Then for any $e > 0$, and $r > r_1$, there exists a U_1, and r_1, such that

$$I(g_r) \leqq I(f)+eU_1.$$

And then
$$I(f_r) \leqq I(g_r) < I(f)+eU_1.$$

In the same manner, if we put

$$h_{r,s}=f_r \cap f_{r+1} \cap \cdots \cap f_{r+s},$$

then
$$h_{r,s} \geqq h_{r,s+1} \geqq \cdots \rightarrow h_r.$$

Therefore $h_r \leqq h_{r+1} \leqq \cdots$, h_r is integrable and $I(f)=\lim I(h_r)$. Thus we have

$$I(h_r) > I(f)-eU_2 \qquad (r \geqq r_2).$$

If we put
$$U_1 \cup U_2 = U,$$

then
$$|I(f_r)-I(f)| < eU.$$

Hence $\lim I(f_n)$ exists and equals to $I(f)$.

Summing up above results, we have reached that $I(f)$ satisfied the conditions (C), (A), (L), (P), and moreover the Lebesgue's convergence theorem (4.7) and the Fatou-Levi's theorem (4.6), where the functions now belong to the class of integrable functions. Thus the integral $I(f)$ becomes to have right to be called Lebesgue integral.