103. On the Congruence Relations on Lattices.

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G. Birkhoff[1] has proved that the congruence relations on any modular lattice of finite dimension form a Boolean algebra. The object of this paper is to prove that the congruence relations on any lattice of finite dimension form a distributive lattice.

By a "congruence relation" on a lattice with operation $\cup$ and $\cap$ is meant a division of its elements into subsets which preserves the univalence of the operations, e. g. makes the subset containing $x \cup y$ depends only on the subset containing x and the subset containing y, and also for $x \cap y$.

A congruence relation θ on any lattice L of finite dimension is determined by its prime quotients which θ annuls. (a/b is said to be annuled when $a \equiv b(\theta)$). We denote by Θ the set of all prime quotients which θ annuls.

Lemma 1. Θ satisfies the following condition. (1) When $a/b \in \Theta$, u/v is any projective quotient of a/b, and p/q is such a prime quotient as $u \geqq p > q \geqq v$, then $p/q \in \Theta$.

Proof. As u/v is a projective quotient of a/b which θ annuls, u/v is also annuled by θ. Then $p = u \cap p \equiv v \cap p = v$, $q = v \cup q \equiv u \cup q = u$, thus $p \equiv q$.

Lemma 2. let Θ be a set of prime quotients of a lattice L of finite dimension, which satisfies the condition (1) of lemma 1. Let us define $x \equiv y(\theta)$ when $x \cup y$ and $x \cap y$ are connected by a set of prime quotients which are elements of Θ. Then θ is a congruence relation on L.

Proof. In the first place θ gives an equivalence relation. For we have evidently reflexive and symmetric relation. It remains to prove transitive relation: $a \equiv b$, $b \equiv c(\theta)$ induce $a \equiv c(\theta)$. In fact $a \cup b \cup c/a \cup b$ is a projective quotient of $b \cup c/(a \cup b) \cap (b \cup c)$, and $b \cup c \geqq (a \cup b) \cap (b \cup c) \geqq b$, and $b \cup c/b$ is annuled by θ; whence $a \cup b \cup c/a \cup b$ is annuled by θ. By hypothesis $a \cup b/a$ is annuled by θ. Thus $a \cup b \cup c/a$ is annuled and then $a \cup c/a$ is annuled, whence $a \equiv c(\theta)$.

Next θ preserves the univalence of the operations, that is $a \equiv b$, $c \equiv d$ induce $a \cup c \equiv b \cup d(\theta)$. To prove this we can assume $a > b$, $c > d$. $a \cup c/b \cup c$ is a transposed quotient of $a/a \cap (b \cup c)$, $a \geqq a \cap (b \cup c) \geqq b$, and then a/b is annuled by θ, thus by (1) $a \cup c/b \cup c$ is annuled. Similarly $b \cup c/b \cup d$ is annuled by θ, and then $a \cup c \equiv b \cup d$.

For two prime quotients p/q and r/s of a lattice of finite dimension, we write $p/q \geqq r/s$ when there exists a quotient u/v which is a projective quotient of p/q and $u \geqq r > s \geqq v$. This definition obviously satisfies the axioms of partial ordering, and we denote by X this

1) G. Birkhoff, Lattice Theory, p. 43.

partially ordered system. We can also partially order the congruence relations on L by defining $\theta \geqq \theta'$ when $x \equiv y(\theta)$ induces $x \equiv y(\theta')$.

Theorem. The congruence relations on any lattice of finite dimension form a distributive lattice, and is isomorphic with B^X, where B is a lattice formed by two elements.

Proof. Lemma 1 and 2 give us one-to-one correspondence between θ and Θ. If we define $\Theta \geqq \Theta'$ by set-inclusion, this correspondence is isomorphic. To Θ corresponds an element $f_\theta(x)$ of B^X such as $\Theta = (p/q : f_\theta(p/q) = 0)$. By the condition (1) and the definition of X this correspondence is also one-to-one and preserves the inclusion relation. Thus the partially ordered system formed by the congruence relations on L is isomorphic with B^X, and so forms a distributive lattice.