124. *On the Zeros of the Riemann Zeta-function.*

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Littlewood[1] proved that the Riemann zeta-function $\zeta(s)$ $(s=\sigma+it)$ has a zero in the domain: $0<\sigma<\infty$, $|t-T|<\dfrac{16}{\log\log\log T}$ $(T\geqq T_0)$. Simple proofs are given by Hoheisel[2], Titchmarsh[3] and Kramaschke[4]. These authors use the Hadamard's three circles theorem in the proof. I will here give a still simpler proof, where I use the Doetsch's three lines theorem[5] in the modified form.

Theorem. $\zeta(s)$ has a zero in the domain: $0<\sigma<\infty$, $|t-T|<\dfrac{\varkappa}{\log\log\log T}$ $\left(T\geqq T(\varkappa)\right)$, *where $\varkappa$ is any positive number greater than $\dfrac{\pi}{4}$.*

Especially we may take $\varkappa=1$.

First we will prove a lemma.

Lemma. Let $f(z)$ be regular and bounded in $|z|<1$ and $K(r)$ be a circle: $|z-(1-r)|=r$ $(0<r\leqq 1)$ *and* $M(r)=\underset{z \text{ on } K(r)}{\operatorname{Max.}}|f(z)|$. *Then*

$$M(r_2)\leqq M(r_1)^{\frac{\frac{1}{r_2}-\frac{1}{r_3}}{\frac{1}{r_1}-\frac{1}{r_3}}}\,M(r_3)^{\frac{\frac{1}{r_1}-\frac{1}{r_2}}{\frac{1}{r_1}-\frac{1}{r_3}}}\qquad(0<r_1<r_2<r_3\leqq 1)\,.$$

Proof. By $s=\dfrac{1}{1-z}$, we map $|z|<1$ on the half-plane $\Re(s)>\dfrac{1}{2}$, then $K(r)$ becomes a line $\Re(s)=\dfrac{1}{2r}$, so that the lemma follows from the Doetsch's three lines theorem[5].

Proof of the theorem.

Suppose that $\zeta(s)$ has no zero in the domain $\varDelta$: $0<\sigma<\infty$, $|t-T|<c=\dfrac{\varkappa}{\log\log\log T}$ $\left(\varkappa>\dfrac{\pi}{4}\right)$, then $\log\zeta(s)$ is regular in $\varDelta$. We map $\varDelta$ on $|z|<1$ by

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1) Littlewood: Two notes on the Riemann zeta-function. Proc. Cambridge Phil. Soc. **22** (1924).

2) Hoheisel: Jahresbericht Schles. Ges. vaterl. Kultur **99**.

3) Titchmarsh: On the Riemann zeta-function. Proc. Cambridge Phil. Soc. **28** (1932).

4) Kramaschke: Nullstellen der Zetafunktion. Deutsche Math. **2** (1937).

5) Doetsch: Über die obere Grenze des absoluten Betrages einer analytischen Funktion auf Geraden. Math. Z. **8** (1920).

$$z=\varphi(s)=\frac{1-2e^{-a(s-iT)}}{1+2e^{-a(s-iT)}}\cdot\frac{2+e^{-a(s-iT)}}{2-e^{-a(s-iT)}}, \tag{1}$$

where $a=\frac{\pi}{2c}=\lambda\log\log\log T\left(\lambda=\frac{\pi}{2\varkappa}<2\right)$.

Then $s=\infty+iT$ corresponds to $z=1$ and

$$1-z=3e^{-a(s-iT)}\big(1+o(1)\big), \tag{2}$$

where $o(1)\to 0$ for $T\to\infty$, uniformly for $\Re(s)=\sigma\geqq\varepsilon>0$. Hence the segment: $\sigma=$const. $(\geqq\varepsilon>0)$, $|t-T|\leqq c$ is mapped on a curve, which lies between two circles:

$$|z-1|=3e^{-a\sigma}\big(1+o(1)\big), \qquad |z-1|=3e^{-a\sigma}\big(1-o(1)\big). \tag{3}$$

If we put

$$z_1=\varphi(1+\varepsilon+iT),\quad z_2=\varphi\left(\frac{1}{2}-\varepsilon+iT\right),\quad z_3=\varphi(\varepsilon+iT)\left(o<\varepsilon<\frac{1}{2}\right), \tag{4}$$

then by (2),

$$1-z_1=3e^{-a(1+\varepsilon)}\big(1+o(1)\big),\qquad 1-z_2=3e^{-a(\frac{1}{2}-\varepsilon)}\big(1+o(1)\big),$$
$$1-z_3=3e^{-a\varepsilon}\big(1+o(1)\big). \tag{5}$$

Let two circles; $C_0:\left|\frac{z-z_1}{1-z_1z}\right|=z_1$ and $C_1:\left|z-\frac{3}{4}\right|=\frac{1}{4}$ meet at ξ, then we have easily

$$|\xi-1|=\frac{1-z_1}{\sqrt{3-4z_1+3z_1^2}}=\frac{1-z_1}{\sqrt{2}}=\frac{3}{\sqrt{2}}e^{-a(1+\varepsilon)}\big(1+o(1)\big). \tag{6}$$

Let $C_1=C_1'+C_1''$, where C_1' is the part of C_1, which lies inside C_0 and C_1'' is the part of C_1, which lies outside C_0. Then by (6), C_1'' is contained in the circle $:|z-1|\leqq 3e^{-a(1+\varepsilon)}\big(1-o(1)\big)$, so that by (3), the image of C_1'' in Δ lies on the right of the line $\Re(s)=1+\varepsilon$. Hence $|\log\zeta(s)|$ is bounded on C_1''. To evaluate $|\log\zeta(s)|$ on C_1', we map $|z|<1$ on $|x|<1$ by $x=\frac{z-z_1}{1-z_1z}$ and put $F(x)=\log\zeta(s)$, then $|F(0)|$ $=|\log\zeta(1+\varepsilon+iT)|$ is bounded for $T\to\infty$ and $\Re\big(F(x)\big)=\log|\zeta(s)|$ $\leqq\log T$ $(T\geqq T_0)$[1].

Hence by Carathéodory's theorem (Math. Ann. **73**), we have in $|x|<z_1$, or in C_0 and hence on C_1',

$$|\log\zeta(s)|=|F(x)|\leqq\frac{2}{1-z_1}\big(\log T+2|F(0)|\big)\leqq e^{2a}\log T \qquad (T\geqq T_0),$$

so that on C_1,

1) Bieberbach: Lehrbuch der Funktionentheorie, II. S. 348.

$$|\log \zeta(s)| \leqq e^{2a} \log T \qquad (T \geqq T_0). \tag{7}$$

We put

$$K_i : \left| z - \frac{1+z_i}{2} \right| = \frac{1-z_i}{2} = r_i \quad (i=1, 2, 3), \quad M_i = \text{Max.} |\log \zeta(s)| \ \text{ on } \ K_i,$$

then by (5),

$$r_1 = \frac{3}{2} e^{-a(1+\varepsilon)} \left(1+o(1)\right), \qquad r_2 = \frac{3}{2} e^{-a\left(\frac{1}{2}-\varepsilon\right)} \left(1+o(1)\right),$$

$$r_3 = \frac{3}{2} e^{-a\varepsilon} \left(1+o(1)\right).$$

Since K_1 is contained in a circle: $|z-1| = 1-z_1 < 3e^{-a\left(1+\frac{\varepsilon}{2}\right)}\left(1-o(1)\right)$, we see by (3), that its image in Δ lies on the right of the line $\Re(s) = 1+\frac{\varepsilon}{2}$[1], so that $M_1 = O(1)$ for $T \to \infty$ and since for $T \geqq T_0$, K_3 is contained in C_1, we have by (7), $M_3 \leqq e^{2a} \log T$.

Hence by the lemma,

$$M_2 \leqq M_1^{\frac{\frac{1}{r_2}-\frac{1}{r_3}}{\frac{1}{r_1}-\frac{1}{r_3}}} M_3^{\frac{\frac{1}{r_1}-\frac{1}{r_2}}{\frac{1}{r_1}-\frac{1}{r_3}}} = M_1^{o(1)} M_3^{1-e^{-a\left(\frac{1}{2}+2\varepsilon\right)}\,(1+o(1))} \leqq$$

$$\text{const.} \log T \, e^{2\lambda \log\log\log T - (\log\log T)^{1-\lambda\left(\frac{1}{2}+2\varepsilon\right)}\,(1+o(1))}.$$

Since $\lambda < 2$, we take ε so small that $1-\lambda\left(\frac{1}{2}+2\varepsilon\right) > 0$, then $M_2 = o(1) \log T$, so that

$$\left|\zeta\left(\frac{1}{2}-\varepsilon+iT\right)\right| = T^{o(1)}, \tag{8}$$

$$\left|\zeta\left(\frac{1}{2}+\varepsilon+iT\right)\right| = T^{o(1)}. \tag{9}$$

From the functional equation of $\zeta(s)$, we have

$$\left|\zeta\left(\frac{1}{2}-\varepsilon+iT\right)\right| = \left|\zeta\left(\frac{1}{2}+\varepsilon+iT\right)\right| \cdot \left|\chi\left(\frac{1}{2}+\varepsilon+iT\right)\right|, \tag{10}$$

where $\left|\chi\left(\frac{1}{2}+\varepsilon+iT\right)\right| \sim \text{const.}\, T^{\varepsilon}$ for $T \to \infty$.

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1) In fact, the image of K_1 in Δ lies on the right of the line $\Re(s) = 1+\varepsilon$. For, since Δ is a convex domain, by Radó's theorem (Math. Ann. **102**) any circle $|z| = r(<1)$ corresponds to a convex curve in Δ and since any circle in $|z| < 1$ can be transformed into a circle of the form $|z| = r(<1)$, its image is also a convex curve. K_1, being the limit of circles in $|z| < 1$, is mapped on a convex curve in Δ. Since the image of K_1 passes through $s = 1+\varepsilon+iT$ and is symmetric to the line $t = \text{const.} = T$, it lies on the right of the line $\Re(s) = 1+\varepsilon$.

From (9), (10), we have $\left|\zeta\left(\frac{1}{2}-\varepsilon+iT\right)\right| \geqq T^{\frac{\varepsilon}{2}}$ $(T \geqq T_0)$, which contradicts (8). Hence $\zeta(s)$ has a zero in the domain: $0<\sigma<\infty$, $|t-T|<\frac{\varkappa}{\log\log\log T}$ $\left(T \geqq T(\varkappa)\right)$, where $\varkappa$ is any positive number greater than $\frac{\pi}{4}$