122. On some Properties of Hausdorff's Measure and the Concept of Capacity in Generalized Potentials.

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I. Hausdorff's measure and upper density.

Let Ω be any *separable* metric space with the distance $\rho(p,q)$ for $p, q \in \Omega$.

A *sphere* in Ω with radius r, of centre a is the set of points p such that $\rho(p,a) < r$.

Given any set $E \subset \Omega$. let $\delta(E)$ be the diameter of E, that is, $\delta(E) = \sup_{p,q \in E} \rho(p,q)$.

Now, let $h(r)$ be a positive, continuous, monotone-increasing function defined for $r > 0$ near the origin such that

$$\lim_{r \to 0} h(r) = 0 .$$

Taking any sequence of spheres $\{S_i\}_{i=1,2,\dots}$ such that

$$\text{(i)} \quad \sum_{i=1}^{\infty} S_i \supset E , \qquad \text{(ii)} \quad \delta(S_i) < \varepsilon \ (i = 1, 2, \dots) ,$$

let us put $m_h(E,\varepsilon) = \inf \sum_{i=1}^{\infty} h[\delta(S_i)]$ for fixed $\varepsilon > 0$, and write $m_h(E) = \lim_{\varepsilon \to 0} m_h(E,\varepsilon)$ which is called h-measure of E. In this definition, we may assume, without loss of generality, that each S_i has points common with E. This measure, introduced first by F. Hausdorff[1] is known to have the property of Carathéodory's outer measure and therefore the measurable class of sets with respect to the h-measure contains all the Borel sets.

Moreover, h-measure is a *regular measure*[2], that is to say, for any set $E \subset \Omega$, there exists a Borel set, $H \in \mathfrak{G}\delta$, such that $H \supset E$ and $m_h(H) = m_h(E)$.

If Ω is 2-dimensional Euclidean space, $m_h(E)$ for $h(r) = \frac{\pi}{4} r^2$, $h(r) = r^\alpha$ ($\alpha > 0$) and $h(r) = \left(\log \frac{1}{r}\right)^{-1}$ are Lebesgue's plane measure, α-dimensional measure (if $\alpha = 1$, then called Carathéodory's linear measure or length of E) and logarithmic measure respectively.

Given a set E and a point $p \in \Omega$, we shall define the *upper density* of E at p with respect to the h-measure by the following expression:

$$\Delta_h(p, E) = \overline{\lim_{\delta(S) \to 0}} \frac{m_h(E \cdot S)}{h[\delta(S)]} ,$$

1) F. Hausdorff. Dimension und äusseres Mass, Math. Annalen., **78** (1919).
2) F. Hausdorff. Loc. cit.

where S is any sphere containing the point p.

II. Density theorem.

We shall prove in this section the following fundamental theorem.

Density theorem: *Let E be any bounded set with finite h-measure. Then the subset H of points $p(\in E)$ which satisfy the condition $\Delta_h(p, E) < 1$ is of h-measure zero.*

Proof. Denoting by H_m the set of points $p \in E$ which satisfy $\Delta_h(p, E) < 1 - m^{-1}$, $(m = 2, 3, \dots)$, we have

$$H = H_2 + H_3 + \cdots .$$

For any point p of H_m for a fixed m, there exists a positive integer n such that for *all* the spheres S containing the point p, with diameters less than n^{-1}, we have

$$(1) \qquad m_h(E \cdot S) < (1 - m^{-1}) h[\delta(S)].$$

Fixing again n for a moment, let us denote by H_{mn} the set of all the points $p(\in H_m)$ which satisfy the condition (1) for all S with $\delta(S) < n^{-1}$ and $S \ni p$. Then, varying n from 1 to ∞, we have evidently

$$H_m = \sum_{n=1}^{\infty} H_{mn} .$$

Hence we have

$$H = \sum_{m=2}^{\infty} H_m = \sum_{m,n} H_{mn} ,$$

and accordingly

$$m_h(H) \leqq \sum_{m,n} m_h(H_{mn}) .$$

Suppose now that our theorem were false. Then we would have $m_h(H) > 0$ and find two positive integers $m = \mu$, $n = \nu$ for which $m_h(H_{\mu\nu}) > 0$.

By the definition of $H_{\mu\nu}$, for any $p \in H_{\mu\nu}$ and for any sphere S containing p, with diameter $< \nu^{-1}$, we have

$$(2) \qquad m_h(S \cdot E) < (1 - \mu^{-1}) h[\delta(S)] .$$

By the relation $m_h(H_{\mu\nu}) > 0$, we can choose a sequence of spheres $\{S_i\}$ such that $\delta(S_i) < \nu^{-1}$, $\sum S_i \supset H_{\mu\nu}$, $H_{\mu\nu} \cdot S_i \neq 0$ and

$$(3) \qquad \sum h[\delta(S_i)] < (1 + \mu^{-1}) m_h(H_{\mu\nu}) .$$

Since each S_i contains points of $H_{\mu\nu}$, we have

$$\begin{aligned} m_h(H_{\mu\nu}) &\leqq \sum_{i=1}^{\infty} m_h(H_{\mu\nu} \cdot S_i) \leqq \sum m_h(E \cdot S_i) \\ &< (1 - \mu^{-1}) \sum h[\delta(S_i)] \qquad \text{(from (2))} \\ &< (1 - \mu^{-1})(1 + \mu^{-1}) m_h(H_{\mu\nu}) , \quad \text{(from (3))} \end{aligned}$$

from which we must have an absurd relation $1 < 1 - \mu^{-2}$ and this completes the proof.

Besides the generality of our theorem, this result is rather unexpected, since if we use "symmetric" upper density for $\Delta_h(p, E)$, that is, in the definition of $\Delta_h(p, E)$, we consider a little narrower class of spheres, of centre p, then our theorem does not hold as stated. Indeed, in case of 2-dimensional Euclidean space and $h(r)=r$, A. S. Besicovitch has shown that there exists a set of finite length at almost every point[1] of which the upper symmetric density is $\frac{1}{2}$.

III. Generalized potentials and capacity.

Let $\Phi(r)$ be a monotone-decreasing, continuous function defined for $0<r<+\infty$ such that $\lim_{r\to 0}\Phi(r)=+\infty$. It is always possible to find a positive number δ_0 such that for $0<r<\delta_0$ we have always $\Phi(r)>0$.

Examples: $\Phi(r)=\frac{1}{r^\alpha}\ (\alpha>0)$, $\Phi(r)=\log\frac{1}{r}$,

$$\Phi(r)=\frac{\left(\log\frac{1}{r}\right)^\beta\left(\log\log\frac{1}{r}\right)^\gamma\cdots}{r^\alpha}\qquad(\alpha,\beta,\gamma,\cdots\geqq 0).$$

Let $\mathfrak{B}$ be the family of all the Borel sets in the metric space Ω considered.

Given a bounded Borel set E, we define a positive mass-distribution μ on E, of total mass $m(>0)$, as the total additive, non-negative set function $\mu(A)\,(A\in\mathfrak{B})$ such that

$$\mu(E)=m \qquad \text{and} \qquad \mu(\Omega-E)=0.$$

The function $\Phi(\rho(p,q))$, regarded as a function of p and q $(p\neq q)$, being continuous and bounded from below for each fixed p, the integral

$$u(p)=\int_\Omega\Phi(\rho(p,q))d\mu(q)=\int_E\Phi(\rho(p,q))d\mu(q)$$

always exists and has the value $\leqq+\infty$ and $>-\infty$.

We call this integral, regarded as a function of $p(\in\Omega)$ the *potential* by the distribution μ, of the function Φ.

Let μ be any positive mass-distribution on E, of total mass 1, and let us put

$$\sup_{p\in\Omega}\int_\Omega\Phi(\rho(p,q))d\mu(q)=V_\mu^\Phi(E).$$

Varying μ, let us consider the lower bound $\inf_\mu V_\mu^\Phi(E)=V^\Phi(E)$.

We define Φ-capacity of E, $C^\Phi(E)$, as follows:

$$\text{if}\quad V^\Phi(E)<+\infty,\quad\text{then}\quad C^\Phi(E)=\Phi^{-1}\{V^\Phi(E)\},$$

$$\text{and if}\quad V^\Phi(E)=+\infty,\quad\text{then}\quad C^\Phi(E)=0.$$

It is evident that Φ-capacity is non-negative and if $E_1\subset E_2$, then $V^\Phi(E_1)\geqq V^\Phi(E_2)$ and $C^\Phi(E_1)\leqq C^\Phi(E_2)$.

1) This means "with a possible exception of a set of h-measure zero". A. S. Besicovitch's example is given in his paper entitled "On the fundamental geometrical properties of linearly measurable plane set of points". Math. Annalen, **98** (1928).

Given an arbitrary set A in Ω, let us define $C^{\phi}(A)$ as $\sup_{E\subset A} C^{\phi}(E)$ for any bounded $E\subset A$, $E\in\mathfrak{B}$.

Then, we have again

$$C^{\phi}(A_1)\leqq C^{\phi}(A_2)$$

whenever $A_1\subset A_2$.

If $C^{\phi}(A)>0$, we can find a bounded Borel set E, contained in A, satisfying the condition $\Phi\big(C^{\Phi}(E)\big)=V^{\phi}(E)<+\infty$, from which we have for some positive mass-distribution μ on E, of total mass 1,

$$u(p)=\int_E \Phi\big(\rho(p,q)\big)d\mu(q)<V^{\Phi}(E)+\varepsilon,$$

where ε is a given positive number. This shows that the potential $u(p)$ is bounded from above.

Conversely, if there exist a bounded Borel set E, contained in A, and a positive mass-distribution μ on E such that the potential $u(p)$ by μ is bounded from above, then we have $C^{\phi}(A)>0$.

For, $\nu=\mu/\mu(E)$ being a positive mass-distribution on E of total mass 1, we have

$$M>u(p)=\mu(E)\int_E \Phi\big(\rho(p,q)\big)d\nu(q),$$

which shows $V_\nu^{\Phi}(E)\leqq M/\mu(E)$ and consequently $V^{\Phi}(E)\leqq M/\mu(E)$. Hence

$$C^{\Phi}(A)\geqq C^{\Phi}(E)\geqq \Phi^{-1}\big(M/\mu(E)\big)>0.$$

Theorem 1. *If* $E_i\in\mathfrak{B}$ *and* $C^{\phi}(E_i)=0$ $(i=1,2,\dots)$, *then* $C^{\phi}(\sum_i E_i)=0$.

Proof. First we prove this theorem under the condition that the set $E=\sum_i E_i$ is bounded and consequently each E_i is also bounded. Suppose that $C^{\phi}(E)>0$.

Then we can find a positive mass-distribution μ on E, of total mass 1, by which the potential $u(p)$ is bounded from above $<M$.

Since $1=\mu(E)\leqq\sum\mu(E_i)$, there exists an integer i for which $\mu(E_i)>0$.

Now we distinguish here two cases.

1°) If $\Phi\big(\rho(p,q)\big)\geqq 0$ for every pair of p and $q\in\Omega$, then

$$\int_{E_i}\Phi\big(\rho(p,q)\big)d\mu(q)\leqq\int_E\Phi\big(\rho(p,q)\big)d\mu(q)<M.$$

This shows $C^{\phi}(E_i)>0$, which is a contradiction.

2°) If $\Phi\big(\rho(p,q)\big)<0$ for some p and q, then there exists a positive number δ_0 such that $\Phi(r)\geqq 0$ for $0<r\leqq\delta_0$ and $\Phi(r)<0$ for $r>\delta_0$.

Let S be a sphere which contains E and S_0 a sphere concentric with S with radius $r_0+\delta_0$, where r_0 is the radius of S.

We have then for $p\notin S_0$ and $E'\subset E$, $\in\mathfrak{B}$

$$\int_{E'} \Phi\big(\rho(p,q)\big)d\mu(q) \leqq 0 . \tag{4}$$

It is obvious that

$$\Phi[\delta(S_0)] < 0 . \tag{5}$$

From $$M > \int_E \Phi\big(\rho(p,q)\big)d\mu(q) = \int_{E_i} + \int_{E-E_i} ,$$

we have $$M - \int_{E-E_i} > \int_{E_i} \Phi\big(\rho(p,q)\big)d\mu(q) .$$

If $p \in S_0$ on the one hand, we see by (5)

$$\int_{E-E_i} \Phi\big(\rho(p,q)\big)d\mu(q) \geqq \Phi[\delta(S_0)] \int_{E-E_i} d\mu(q) \geqq \Phi[\delta(S_0)] \int_E d\mu(q)$$
$$= \Phi[\delta(S_0)] ,$$

from which follows

$$M - \Phi[\delta(S_0)] > \int_{E_i} \Phi\big(\rho(p,q)\big)d\mu(q) .$$

If $p \notin S_0$ on the other hand, we see from (4)

$$0 \geqq \int_{E_i} \Phi\big(\rho(p,q)\big)d\mu(q) .$$

Then we have for every $p \in \Omega$

$$\mathrm{Max}\,\big(M - \Phi[\delta(S_0)], 0\big) \geqq \int_{E_i} \Phi\big(\rho(p,q)\big)d\mu(q) ,$$

which shows that $C^{\Phi}(E_i) > 0$, hence a contradiction.

Next we shall show that the theorem holds true even if $\sum E_i = E$ is not bounded.

Let E' be any bounded Borel set contained in E. We have only to prove that $C^{\Phi}(E') = 0$.

Put $E' \cdot E_i = E'_i (\subset E_i)$ and we find that these are Borel sets, whose sum is a bounded Borel set E'.

But $C^{\Phi}(E_i) = 0$, hence $C^{\Phi}(E'_i) = 0$ and this shows, from what is proved above, that $E' = \sum_i E'_i$ is of Φ-capacity zero, which establishes our proof completely.

Theorem 2. *Given any positive mass-distribution μ on a bounded Borel set E, then the set E_1 of points at which the potential $u(p)$ by μ has the value $+\infty$ is of Φ-capacity zero.*

Proof. As is easily seen, $\mu(p)$ is a semi-continuous function and the set E_1 is a Borel set. Morever, since every point of E_1 is a limiting point of E, E_1 is also bounded.

If $C^{\Phi}(E_1) > 0$, there would exist a positive mass-distribution ν on E_1 for which

$$\int_{\Omega} \Phi\big(\rho(p,q)\big)d\nu(p) = \int_{E_1} \Phi\big(\rho(p,q)\big)d\nu(p) < M(< +\infty) .$$

But $$\int_{E_1} u(p)d\nu(p)=\int_\Omega\Big(\int_\Omega \Phi\big(\rho(p,q)\big)d\mu(q)\Big)d\nu(p)$$
$$=\int_\Omega\Big(\int_\Omega \Phi\big(\rho(p,q)\big)d\nu(p)\Big)d\mu(q)$$

by Tonelli's theorem[1] and from this we have

$$\int_{E_1} u(p)d\nu(p)\leqq M\int_\Omega d\mu(q)=M\cdot\mu(E)<+\infty\,,$$

which shows that at least at one point of E_1, $u(p)$ must not take the value $+\infty$, and this disagrees with the definition of E_1.

IV. Relations between capacity and h-measure.

The object of this section is to prove the following important result.

Theorem A. *If* $C^\Phi(A)>0$, *then* $m_h(A)=+\infty$, *where* $h(r)=[\Phi(r)]^{-1}$. *That is to say, if* $m_h(A)<+\infty$, *then* $C^\Phi(A)=0$.

This theorem, in case of 2-dimensional Euclidean plane and the special function $\Phi(r)=\log\frac{1}{r}$, had been conjectured by R. Nevanlinna and was proved by P. Erdös and J. Gillis after complicated calculations, and, in case of Euclidean plane and $\Phi(r)=\frac{1}{r}$, has been proved by T. Ugaheri[2].

The proof of our theorem depends essentially upon our Density Theorem given in section II.

Let us begm with some definitions.

Denoting by $S(p,r)$ the sphere of centre p with radius r, we see at once that $m_h\big(E\cdot S(p,r)\big)$ can not increase as r decreases and put

$$o_h(p,E)=\lim_{r\to 0} m_h\big(E\cdot S(p,r)\big)\,.$$

Lemma 1. *If* $m_h(E)<+\infty$, *then the set* H *of all the points* $p(\epsilon E)$ *for which* $o_h(p,E)>0$ *is enumerable and consequently, of h-measure zero.*

Proof. Let H_n be the set of points $p(\epsilon E)$ for which

$$n^{-1}<o_h(p,E)\,.$$

Then $$H=\sum_{n=1}^{\infty}H_n\,.$$

We see that each H_n is a finite set. For, supposing the contrary, there would exist m different points $p_1, p_2, \cdots, p_m$ of H_n, m being an integer $>n\cdot m(E)$.

1) S. Saks. Théorie de L'integrale. (1933) p. 75, but in a restricted form, which can be modified easily in the general form. We remark here that the function $\Phi(\rho(p,q))$ for $p\epsilon E_1$ and $q\epsilon E$ is bounded from below since E and E_1 are bounded, so that Tonelli's theorem can be applicable.

2) T. Ugaheri. Proc. **18** (1942), 602.

P. Erdös and J. Gillis. Note on the Transfinite Diameter, Journ. of Lond. Math. Soc. Vol. **12** (1937).

Then there exist m spheres $S(p_i, r_i)(i=1, 2, \dots, m)$ each of which lies outside the others. Hence

$$m_h(E) \geqq m_h\Big(E \cdot \sum_{i=1}^{m} S(p_i, r_i)\Big) = \sum_{i=1}^{m} m_h\Big(E \cdot S(p_i, r_i)\Big)$$
$$> m \cdot n^{-1} > m_h(E),$$

which is absurd.

Lemma 2. *If E is a bounded Borel set and $C^{\Phi}(E) > 0$, then $m_h(E) > 0$, where $\Phi = [h]^{-1}$ near the origin.*

Proof. Take δ_0 such that $\Phi(r) > 0$ for $0 < r < \delta_0$. Let $\{S_i\}$ be any sequence of spheres satisfying the following conditions

$$\delta(S_i) < \delta_0 \quad \text{and} \quad E \subset \sum_{i=1}^{\infty} S_i .$$

Then we have $0 < C^{\Phi}(E) = C^{\Phi}(\sum E \cdot S)$. Hence there is an integer $i = i_0$ for which $C^{\Phi}(ES_{i_0}) > 0$, for, otherwise, we would have $C^{\Phi}(E) = 0$ by Theorem 1. Hence there exists a positive mass-distribution μ on $E_0 = E \cdot S_{i_0}$ such that

$$u(p) = \int_{E_0} \Phi\big(\rho(p, q)\big) d\mu(q) < M,$$

where M is a positive constant.

Let $\{S_0^i\}_{i=1,2,\dots}$ be any sequence of spheres satisfying the following conditions:

$$\sum_{i=1}^{\infty} S_0^i \supset E_0, \quad S_0^i \cdot E \neq 0 \quad \text{and} \quad \delta(S_0^i) < \delta_0 .$$

Since each $S_0^i \cdot E_0$ is not empty, let p_i be any one of the points belonging to $S_0^i \cdot E_0$. Now, as $\Phi\big(\rho(p, q)\big) > 0$ for every $p, q \in E_0$, we have evidently

$$\int_{S_0^i E_0} \Phi\big(\rho(p_i, q)\big) d\mu(q) \leqq \int_{E_0} \Phi\big(\rho(p_i, q)\big) d\mu(q) < M$$

and

$$\int_{S_0^i E_0} \Phi\big(\rho(p_i, q)\big) d\mu(q) \geqq \int_{S_0^i E_0} \Phi[\delta(S_0^i)]\, d\mu(q) = \frac{\mu(S_0^i \cdot E_0)}{h[\delta(S_0^i)]} ,$$

or combining both,

$$M^{-1} \cdot \mu(E_0 \cdot S_0^i) < h[\delta(S_0^i)] \qquad (i = 1, 2, \dots),$$

and summing up,

$$M^{-1}\mu(E_0) \leqq M^{-1} \sum \mu(E_0 \cdot S_0^i) < \sum h[\delta(S_0^i)],$$

in which, taking the lower bound of the right hand-side, we have

$$M^{-1}\mu(E_0) \leqq m_h(E_0, \delta_0) .$$

But since $\mu(E_0) > 0$ and $m_h(E_0, \delta_0) \leqq m_h(E_0) \leqq m_h(E)$, it follows immediately

$$0 < m_h(E) .$$

Proof of Theorem A. Choose δ_0 so that $\Phi(r)>0$ for $0<r\leqq\delta_0$.

We have only to prove that if $m_h(A)<+\infty$, then, for any bounded Borel set $E\subset A$, $C^{\Phi}(E)=0$.

Now, if $m_h(E)=0$, then by Lemma 2, we have $C^{\Phi}(E)=0$, so that we may suppose $0<m_h(E)<+\infty$, by which we may consider m_h a positive mass-distribution on E.

Let E_1 and E_2 be the sets of the points $(\in E)$ which satisfy $o_h(p, E)>0$ and $\Delta_h(p, E)<1$ respectively.

Then E_1+E_2 is of h-measure zero by Lemma 1, 2 and Density Theorem. By the regularity of h-measure, there exists a bounded Borel set $H\supset E_1+E_2$ whose h-measure is also zero.

Then, for each point $p\in E_0=E-H$,

$$\Delta_h(p, E)\geqq 1 \tag{6}$$

and

$$o_h(p, E)=0 \quad \text{or} \quad \lim_{r\to 0} m_h\big(E\cdot S(p, r)\big)=0\,. \tag{7}$$

By (6), we can choose a sequence $\{S_i\}$ of spheres which satisfy the following conditions:

$$S_i\ni p, \quad \delta(S_i)\to 0\,(i\to\infty) \qquad \text{and}$$

$$\frac{m_h(E\cdot S_i)}{h[\delta(S_i)]}>\frac{1}{2} \qquad (i=1, 2, \dots)\,. \tag{8}$$

Since $S_i\ni p$ and $\delta(S_i)\to 0$, to each positive $r(\leqq\delta_0)$, there exists a positive integer N such that for $i>N$, we have

$$S_i\subset S(p, \delta)\,.$$

As $\Phi\big(\rho(p, q)\big)>0$ for every $q\in S(p, \delta)$, we have for $i>N$

$$\int_{E\cdot S(p,r)}\Phi\big(\rho(p, q)\big)dm_h(q)\geqq\int_{E\cdot S_i}\Phi\big(\rho(p, q)\big)dm_h(q) \tag{9}$$

$$\geqq\Phi[\delta(S_i)]m_h(E\cdot S_i)=\frac{m_h(E_i\cdot S)}{h[\delta(S_i)]}>\frac{1}{2} \qquad \text{(by (8))}.$$

We shall now show $u_h(p)=\int_E\Phi\big(\rho(p, q)\big)dm_h(q)=+\infty$ at $p\in E_0$
If $u_h(q)<+\infty$, then it would follow

$$\int_{E\cdot S(p,\,\delta_0)}\Phi\big(\rho(p, q)\big)dm_h(q)<+\infty\,,$$

that is to say,

$$\int_{E\cdot S(p,\,\delta_0)}\Phi\big(\rho(p, q)\big)dm_h(q)$$

would exist and be finite, since

$$\int_{E\cdot S(p,\,\delta_0)}=\int_E-\int_{E-S(p,\delta_0)}\Phi\big(\rho(p, q)\big)dm_h(q)$$

where the last term is evidently bounded.

From the existence of the finite integral $\int_{E\cdot S(p,\delta_0)} \Phi\big(\rho(p,o)\big)dm_h(q)$, it follows that the indefinite integral

$$F(e)=\int_e \Phi\big(\rho(p,q)\big)dm_h(q)$$

is, with respect to m_h, an absolutely continuous set-function of Borel sets e contained in $E\cdot S(p,\delta_0)$.

But according to (7), $\lim_{r\to 0} F\big(E\cdot S(p,r)\big)=0$, which would contradict (9).

Thus, we have seen that at every point p of $E_0=E-H$, the potential $u_h(p)$ takes the value $+\infty$, from which follows by Theorem 2 that $C^{\Phi}(E_0)=0$ as E_0 is, as well as E and H, a bounded Borel set, while the set H is, as shown, of h-measure zero by Lemma 2.

The set E is, as the sum of two Borel sets of Φ-capacity zero, also of Φ-capacity zero and our Theorem ist completely proved.

Noticing that h-measure is regular, we have by Theorem A and 1 the following:

Theorem B. *If* $A=\sum_{i=1}^{\infty} A_i$, $m_h(A_i)<+\infty$ $(i=1,2,\ldots)$, *then* $C^{\Phi}(A)=0$ *where* $[\Phi]^{-1}=h$.

V. A Problem.

It would be interesting if one could make clear under what conditions two special equivalent metrics keep sets of Φ-capacity zero invariant with each other.