2. *On the Axioms of the Theory of Lattice.*

By Masatada KOBAYASI.

Mathematical Seminary, Hokkaido Imperial University, Sapporo.

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1. System $\sum$.

Suppose that there is a set S of elements, between each two of which two dualistic operations $\cup$ and $\cap$ are so defined that their results are unique and belong to S. If the two operations satisfy the following postulates:

L 1.	Idempotent law:	(a)	$x\cup x=x$	for all x,
		(b)	$x\cap x=x$	for all x,
L 2.	Commutative law:	(a)	$x\cup y=y\cup x$	for all x and y,
		(b)	$x\cap y=y\cap x$	for all x and y,
L 3.	Associative law:	(a)	$x\cup(y\cup z)=(x\cup y)\cup z$	for all x, y and z,
		(b)	$x\cap(y\cap z)=(x\cap y)\cap z$	for all x, y and z,
L 4.	Absorptive law:	(a)	$x\cup(x\cap y)=x$	for all x and y,
		(b)	$x\cap(x\cup y)=x$	for all x and y,

then the set S is called a lattice for the operations $\cup$ and $\cap$[1].

G. Köthe and H. Hermes showed that if L 4 is satisfied then L 1 does so (Enzyklopädie Bd. I–1 Heft 5, 13 (1939), and they took L 2–L 4 as the axioms for the lattice.

Now, we replace the Absorptive law L 4 by a weaker postulate, viz.[2],

L 4* (a) if $y\cup x=x$ then $y\cap x=y$,

(b) if $y\cap x=x$ then $y\cup x=y$;

and this, together with L 1 (a), L 2, L 3 will be taken as postulates for a "System $\sum$"[3].

We shall demonstrate the independency of postulates of $\sum$. Before doing so, we enumerate some relations between these postulates.

(I) L 1 (a) *and* L 4 (a) *imply* L 1 (b). In fact, by L 1 (a),

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1) Cf. Ore, *On the Foundation of Abstract algebra 1*, Ann. Math., **36**, 409 (1935), Philip M. Whitmann, *Free lattices*, Ann. Math., **42**, 325 (1941), and G. Birkhoff, *Lattice theory*, Ammer. Math. Soc. Coll. Pub. XXV (1940), etc.

2) In the case when L 2 holds, we may use instead of L 4* (a) any one of the following three postulates: (1) *if* $x\cup y=x$ *then* $y\cap x=y$, (2) *if* $y\cap x=x$ *then* $x\cap y=y$, (3) *if* $x\cup y=x$ *then* $x\cap y=y$, and also instead of L 4 (b) any one of the three postulates: (1) if $x\cap y=x$ then $y\cup x=y$, (2) if $y\cap x=x$ then $x\cup y=y$, (3) if $x\cap y=x$ then $x\cup y=y$.

3) In this "System $\sum$", we may use the postulate L 1 (b), instead of L 1 (a). Cf. (1).

Then by Footnote 2) there are 32 equivalent systems of postulates for a lattice.

$x \cup x = x$, thus by L 4* (a) $x \cap x = x$. Similarly, L 1 (b) *and* L 4* (b), *imply* L 1 (a). Hence, when L 4* hold L 1 (a) equivalent with L 1 (b)[1]

(II) L 1 (a), L 2 (a), L 3 (a), and L 4* (a) *imply* L 4 (b), and L 1 (b), L 2 (b), L 3 (b), *and* L 4* (b) *imply* L 4 (a). In fact, by L 3 (a) $x \cup (x \cup y) = (x \cup y) \cup y$, thus by L 1 (a) $x \cup (x \cup y) = x \cup y$, hence by L 4 (a) $x \cap (x \cup y) = x$.

(III) *The two following classes of postulates are identical:*
(i) *the class of* L 1 (a), L 2, L 3, and L 4*;
(ii) *the class of* L 2, L 3, L 4.

2. Independency of postulates of the System $\sum$.

Example 1. Let S be a set of zero and positive integers. Let

$$x \cup y = \begin{cases} x+y, & \text{if } x \neq 1 \text{ and } y \neq 1, \\ 1, & \text{if } x = 1 \text{ or } y = 1, \end{cases}$$

and let $x \cap y = xy$.

Then postulates L 2 (a), L 2 (b), and L 3 (b) are obviously satisfied while L 1 (a) is not. In this system,

$$x \cup (y \cup z) = \begin{cases} x+y+z, & \text{if } x \neq 1,\ y \neq 1, \text{ and } z \neq 1 \\ 1, & \text{if } x = 1,\ y = 1, \text{ or } z = 1, \end{cases}$$

and

$$(x \cup y) \cup z = \begin{cases} x+y+z, & \text{if } x \neq 1,\ y \neq 1, \text{ and } z \neq 1 \\ 1, & \text{if } x = 1,\ y = 1, \text{ or } z = 1. \end{cases}$$

Hence L 3 (a) is satisfied. Again, $y \cup x = x$, if and only if $x = 1$ or $y = 0$, while $y \cap 1 = y$, and $0 \cap x = 0$. And $y \cap x = x$, if and only if $y = 1$ or $x = 0$, while $1 \cup x = 1$, and $y \cup 0 = y$. Hence L 4* (a) and L 4* (b) are satisfied.

Example 2. Let S be the set of the three elements and a, b, and c, and let $x \cup y$ and $x \cap y$ be defined as in the following tables:

$x \cup y$

$x \backslash y$	a	b	c
a	a	a	a
b	a	b	a
c	a	a	c

$x \cap y$

$x \backslash y$	a	b	c
a	a	a	a
b	b	b	b
c	c	c	c

Then postulates L 1 (a) and L 2 (a) are plainly satisfied while L 2 (b) does not hold. In this system, $y \cup x = x$, if and only if

(i) $x = a$ and $y = a$, (ii) $x = a$ and $y = b$,

(iii) $x = a$ and $y = c$, (iv) $x = b$ and $y = b$,

or (v) $x = c$ and $y = c$.

1) (I) asserts that L 1 (b) as postulate is superfluous for the "system $\sum$".

Therefore (i) $a \frown a=a$, (ii) $b \frown a=b$, (iii) $c \frown a=c$, (iv) $b \frown b=b$, and (v) $c \frown c=c$, L 4* (a) is satisfied.

Also, $y \frown x=x$, if and only if (i) $x=a$ and $y=a$, (ii) $x=b$ and $y=b$, or (iii) $x=c$ and $y=c$, then by L 1 (b) which obtains by (1), L 4 (b) hold. Since

$$x \smile (y \smile z)=\begin{cases} a & \text{if } x=a,\ y=a,\ \text{or } z=a, \\ a & \text{if } x \neq a,\ y \neq a,\ \text{and } z \neq a,\ \text{and } x \neq y, \\ & \quad y \neq z,\ \text{or } x \neq z, \\ x & \text{if } x \neq a,\ y \neq a,\ \text{and } z \neq a,\ \text{and} \\ & \quad x=y=z, \end{cases}$$

and

$$(x \smile y) \smile z=\begin{cases} a & \text{if } x=a,\ y=a,\ \text{or } z=a, \\ a & \text{if } x \neq a,\ y \neq a,\ \text{and } z \neq a,\ \text{and } x \neq y, \\ & \quad y \neq z,\ \text{or } x \neq z \\ x & \text{if } x \neq a,\ y \neq a,\ \text{and } z \neq a,\ \text{and} \\ & \quad x=y=z, \end{cases}$$

L 3 (a) is satisfied, and since

$x \frown (y \frown z)=x$ and $(x \frown y) \frown z=x$, L 3 (b) hold.

By the obvious duality between $\smile$ and $\frown$ can construct a model which satisfies L 1 (a), L 2 (b), L 3 (a), L 8 (b), L 4 (a), L 4 (b), but not L 2 (a).

Example 3. Let S be the set of the rational numbers $x \geqq 1$, in which we define the operations $\smile$ and $\frown$ as follows:

$$x \frown y=\begin{cases} x, & \text{if } x=y, \\ 1, & \text{if } x \neq y, \end{cases}$$

and

$$x \frown y=\begin{cases} x & \text{if } y=1 \\ y & \text{if } x=1 \text{ and } y \neq 1, \\ \dfrac{x+y}{2} & \text{if } x \neq 1 \text{ and } y \neq 1. \end{cases}$$

Then L 1 (a), L 2 (b), and L 2 (a) are plainly satisfied, and so are L 3 (a), L 4* (a), and L 4* (b) are satisfied, as

$$x \smile (y \smile z)=\begin{cases} x, & \text{if } x=y=z, \\ 1, & \text{otherwise}; \end{cases}$$

$$(x \smile y) \smile z=\begin{cases} x, & \text{if } x=y=z, \\ 1, & \text{otherwise}, \end{cases}$$

so that $x \smile (y \smile z)=(x \smile y) \smile z$.

In this model, $y \smile x=x$, if and only if (i) $x=y$, or (ii) $x=1$, and $y \neq 1$, and then (i) $x \frown x=x$ and (ii) $y \frown 1=y$. Also $y \frown x=x$, if and only if (i) $y=1$ and $x=1$, (ii) $y=1$ and $x \neq 1$ or (iii) $x \neq 1$, $y \neq 1$

and $x=y$, and then (i) $1 \cup 1=1$. (ii) $1 \cup x=1$, and (iii) $x \cup x=x=y$. Hence L 3 (a), L 4* (a), and L 4* (b) hold. But L 3 (b) is not satisfied, since if $x=2$, $y=4$, and $z=4$, we have $(x \cap y) \cap z=\frac{7}{2}$ and $x \cap (y \cap z)=3$.

Similarly, by duality we obtain *a* set, which satisfies L 1 (a), L 2 (a), L 2 (b), L 3 (b), L 4* (a), and L 4* (b), but not L 3 (a).

Example 4. Let S be a set ef positive integers, and let

$$x \cup y= \max (x, y).$$

$$x \cap y=\text{g. c. d.} (x, y).$$

Then, $y \cap x=x$, if and only if $y=nx$, where n is a positive integer, and since $y \cup x=\max (nx, x)=nx=y$, we get by putting $y=6$ and $x=10$, $y \cup x=x=10$, while $y \cap x=\text{g. c. d.} (6, 10)=2 \neq y$. Hence 14* (b) is satisfied and L 4* (a) in not. This system obviously satisfies L 1 (a), L 2 (a), L 2 (b), L 3 (a), and L 3 (b).

By duality, we can construct a model, which satisfies the postulates of the system $\sum$, except L 4* (b), which is not satisfied.

These examples show that the postulates of $\sum$ are independent.