39. *Notes on Infinite Product Measure Spaces, II.*

By Shizuo KAKUTANI.

Mathematical Institute, Osaka Imperial University.

(Comm. by T. TAKAGI, M.I.A., April 12, 1943.)

1. Let $\{(\Omega^\gamma, \mathfrak{B}^\gamma, m^\gamma) \mid \gamma \in \Gamma\}$ be a family of measure spaces satisfying $m^\gamma(\Omega^\gamma)=1$ for each $\gamma \in \Gamma$. In the first paper[1] we have shown that there exists an independent infinite product measure space $(\Omega^*, \mathfrak{B}^*, m^*)=P\otimes_{\gamma\in\Gamma}(\Omega^\gamma, \mathfrak{B}^\gamma, m^\gamma)$ of these measure spaces. We have proved the existence of the measure $m^*(B^*)$ on the Borel field $\mathfrak{B}^*$ without assuming any topology on the spaces Ω^γ and without using any notion of topology in the proof. In the present paper, however, we shall discuss the case when each space Ω^γ, and hence the infinite product space Ω^*, is a compact Hausdorff space. These assumptions on the spaces Ω^γ will make it possible to discuss the properties of the independent product measure space in more detail. We shall see that there usually exist two infinite independent product measure spaces $(\Omega^*, \mathfrak{B}^*, m^*)$ and $(\Omega^*, \mathfrak{B}^{**}, m^{**})$ defined on the same product space Ω^*, of which the latter is an extension of the former and which usually do not coincide. We shall discuss the conditions under which the completions of these measure spaces coincide, and the results thus obtained will find some applications to the theory of Haar measures on non-separable locally compact topological groups. It is also to be noted that the idea of defining the measure for every open, and hence for every Borel, subset of the product space Ω^* has important consequences in the theory of continuous stochastic processes and of Brownian motions. We shall, however, not enter into these problems in this note, and the discussions of details of the applications are left to another occasion.

2. We begin with preliminary remarks. Let Ω be a compact Hausdorff space, and let $\mathfrak{B}_\Omega$ be the Borel field of subsets B of Ω which is generated by the family $\mathfrak{O}_\Omega$ of all open subsets O of Ω. A subset B of Ω which belongs to $\mathfrak{B}_\Omega$ is called a *Borel subset* of Ω. A countably additive measure $m(B)$ defined on $\mathfrak{B}_\Omega$ (satisfying $m(\Omega)<\infty$) is *regular* if there exists for any $B\in\mathfrak{B}_\Omega$ and for any $\varepsilon>0$ an $O\in\mathfrak{O}_\Omega$ such that $B\subseteqq O$ and $m(O)<m(B)+\varepsilon$.

Let us further denote by $C(\Omega)$ the Banach space of all bounded real-valued continuous functions $x(\omega)$ defined on Ω with $\|x\|=\sup_{\omega\in\Omega}|x(\omega)|$ as its norm. $C(\Omega)$ is at the same time a real normed ring with respect to the ordinary operation of product, and may also be considered as a Banach lattice if we put

$$(1)\qquad x\geqq y \quad \text{if and only if} \quad x(\omega)\geqq y(\omega) \quad \text{for all} \quad \omega\in\Omega.$$

In fact, $C(\Omega)$ is a so-called (M)-space with respect to this partial

1) S. Kakutani, Notes on Infinite Product Measure Spaces, I, Proc. **19** (1943), 148.

ordering. Finally, a bounded linear functional $f(x)$ defined on $C(\Omega)$ is *positive* if

(2) $$f(x) \geqq 0 \quad \text{whenever} \quad x \geqq 0.$$

Theorem 1. Let Ω be a compact Hausdorff space. Then, for any countably additive measure $m(B)$ defined on the Borel field $\mathfrak{B}_\Omega$ with $m(\Omega)=1$, the Radon-Stieltjes integral

(3) $$f(x)=\int_\Omega x(\omega)m(d\omega)$$

defines a positive bounded linear functional $f(x)$ on the Banach lattice $C(\Omega)$, satisfying $\|f\|=m(\Omega)$. Conversely, for any positive bounded linear functional $f(x)$ defined on $C(\Omega)$, there exists a countably additive measure $m(B)$ defined on the Borel field $\mathfrak{B}_\Omega$ such that the relation (3) *is true for any $x(\omega)\in C(\Omega)$. Such a measure $m(B)$ is uniquely determined if we require that $m(B)$ is regular.*

The proof of this theorem is omitted[2].

3. Let now $\{(\Omega^\gamma, \mathfrak{B}^\gamma, m^\gamma) \mid \gamma\in\Gamma\}$ be a family of measure spaces in which each Ω^γ is a compact Hausdorff space, $\mathfrak{B}^\gamma$ is the Borel field $\mathfrak{B}_{\Omega^\gamma}$ of all Borel subsets B^γ of Ω^γ, and $m^\gamma(B^\gamma)$ is a regular countably additive measure defined on $\mathfrak{B}^\gamma$ satisfying $m^\gamma(\Omega^\gamma)=1$ for each $\gamma\in\Gamma$. The family $\mathfrak{O}_{\Omega^\gamma}$ of all open subsets O^γ of Ω^γ is denoted by $\mathfrak{O}^\gamma$. The product space $\Omega^*=P_{\gamma\in\Gamma}\Omega^\gamma$ is then a compact Hausdorff space with respect to the ordinary weak topology of the Cartesian product. A defining neighborhood system of Ω^* is given by the family $\mathfrak{V}^*$ of all open subsets V^* of Ω^* of the form:

(4) $$V^*=O^{\gamma_1}\times\cdots\times O^{\gamma_n}\times P_{\gamma\in\Gamma-\{\gamma_1,\dots,\gamma_n\}}\Omega^\gamma$$

where $O^{\gamma_i}\in\mathfrak{O}^{\gamma_i}$, $i=1,\dots,n$, and $\{\gamma_1,\cdots,\gamma_n\}$ is an arbitrary finite system from Γ. Let us denote by $\mathfrak{B}^*$ the Borel field of subsets of Ω^* generated by $\mathfrak{V}^*$, and by $\mathfrak{O}^*$ the family of all open subsets O^* of Ω^* belonging to $\mathfrak{B}^*$. It is then easy to see that $\mathfrak{O}^*$ and hence $\mathfrak{B}^*$ consist only of those subsets of Ω^* which are determined by a countable number of coordinates[3], while the converse is not necessarily true.

Then, by the results obtained in the first paper, there exists a countably additive measure $m^*(B^*)$ defined on $\mathfrak{B}^*$ such that

(5) $$m^*(V^*)=m^{\gamma_1}(O^{\gamma_1})\times\cdots\times m^{\gamma_n}(O^{\gamma_n})$$

if $V^*\in\mathfrak{V}^*\subseteqq\mathfrak{B}^*$ is of the form (4).

Let us further denote by $\mathfrak{O}^{**}=\mathfrak{O}_{\Omega^*}$ the family of all open subsets O^{**} of Ω^*, and by $\mathfrak{B}^{**}=\mathfrak{B}_{\Omega^*}$ the Borel field of all Borel subsets B^{**} of Ω^*, i.e. the Borel field generated by $\mathfrak{O}^{**}$. If Γ is countable and

2) Cf. S. Kakutani, Concrete representation of abstract (M)-spaces, and the characterization of the space of continuous functions, Annals of Math., **42** (1941).

3) A subset A^* of Ω^* is *determined* by a countable number of coordinates if there exists a countable set $\Gamma_0=\{\gamma_n \mid n=1,2,\dots\}\subseteqq\Gamma$ such that $\omega^*=\{\omega^\gamma \mid \gamma\in\Gamma\}$ belongs to A^* whenever there exists an $\omega_0^*=\{\omega_0^\gamma \mid \gamma\in\Gamma\}$ belonging to A^* such that $\omega^{\gamma_n}=\omega_0^{\gamma_n}$, $n=1,2,\dots$.

if each Ω^γ is compact and metric then it is easy to see that we have $\mathfrak{O}^*=\mathfrak{O}^{**}$ and hence $\mathfrak{B}^*=\mathfrak{B}^{**}$. But these equalities do not hold in general; this is, for example the case if Γ is not countable and if each Ω^γ is a compact metric space containing at least two points.

Theorem 2. There exists a countably additive measure $m^{**}(B^{**})$ *defined on the Borel field* $\mathfrak{B}^{**}=\mathfrak{B}_{\Omega^*}$ *which coincides with* $m^*(B^*)$ *on* $\mathfrak{B}^*$.

Proof. Let us put

$$m^{**}(O^{**})=\sup_{O^*\in\mathfrak{O}^*,\, O^*\leq O^{**}} m^*(O^*) \tag{6}$$

for any open subset O^{**} of Ω^* and

$$m^{**}(A^{**})=\inf_{O^{**}\in\mathfrak{O}^{**},\, O^{**}\geq A^{**}} m^{**}(O^{**}) \tag{7}$$

for any subset A^{**} of Ω^*. It is then clear that $m^{**}(O^*)=m^*(O^*)$ if O^* belongs to $\mathfrak{O}^*$, and that the definition (7) coincides with (6) if $A^{**}=O^{**}$ is open. It is then not difficult to see that $m^{**}(A^{**})$ is a Carathéodory measure and that every open, and hence every Borel, subset of Ω^* is m^{**}-measurable. We omit the detail of the proof[4]. This shows the possibility of extension. It is to be remarked that this extension is unique if we require that the measure $m^{**}(B^{**})$ is regular.

4. In this section we shall give another proof to Theorem 2. Let us consider each $C(\Omega^\gamma)$ as a subring of the real normed ring $C(\Omega)$ by identifying a function $x^\gamma(\omega^\gamma)\in C(\Omega^\gamma)$ with $x^*(\omega^*)=x^\gamma\big(\omega^\gamma(\omega^*)\big)\in C(\Omega^*)$, where $\omega^\gamma(\omega^*)$ is the γ-coördinate of $\omega^*=\{\omega^\gamma\mid\gamma\in\Gamma\}\in\Omega^*$. Let further R be the algebraic subring of $C(\Omega^*)$ generated by the system of subgings $\{C(\Omega^\gamma)\mid\gamma\in\Gamma\}$. R is the set of all functions $y^*(\omega^*)\in C(\Omega^*)$ of the form:

$$y^*(\omega^*)=\textstyle\sum_{k=1}^{p}\prod_{i=1}^{n_k} x^{\gamma_i^k}(\omega^{\gamma_i^k}) \tag{8}$$

where $\big\{\{\gamma_1^k,\dots,\gamma_{n_k}^k\}\mid k=1,\dots,p\big\}$ is a finite system of finite systems from Γ. We shall first see that R is dense in $C(\Omega^*)$. This follows easily from the fact that for any two different points $\omega_1^*,\omega_2^*\in\Omega^*$, there exists a coordinate $\gamma\in\Gamma$ such that $\omega^\gamma(\omega_1^*)\neq\omega^\gamma(\omega_2^*)$ and hence a function $x^\gamma(\omega^\gamma)\in C(\Omega^\gamma)$ such that $x^\gamma\big(\omega^\gamma(\omega_1^*)\big)\neq x^\gamma\big(\omega^\gamma(\omega_2^*)\big)$.

Let now $f^\gamma(x^\gamma)$ be a positive bounded linear functional of norm 1 defined on $C(\Omega^\gamma)$ which corresponds to the measure $m^\gamma(B^\gamma)$, $\gamma\in\Gamma$, by Theorem 1. Let us further put

$$f^*(y^*)=\textstyle\sum_{k=1}^{p}\prod_{i=1}^{n_k} f^{\gamma_i^k}(x^{\gamma_i^k}) \tag{9}$$

if $y^*(\omega^*)$ is of the form (8). Then $f^*(y^*)$ is a positive bounded linear functional defined on R satisfying $\|f^*\|=\sup_{|y^*|\leq1}|f^*(y^*)|=1$. This follows easily from the fact that it is possible to define an independent product measure space $P\otimes_{\gamma\in\Gamma_0}(\Omega^\gamma,\mathfrak{B}^\gamma,m^\gamma)$ on the finite product space $P_{\gamma\in\Gamma_0}\Omega^\gamma$ where $\Gamma_0=\{\gamma_i^k\mid i=1,\dots,n_k;\ k=1,\dots,p\}$.

4) Cf. S. Kakutani, loc. cit. 2).

Since R is dense in $C(\Omega^*)$, the positive bounded linear functional $f^*(y^*)$ defined on R by (9) can be uniquely extended to a positive bounded linear functional $f^*(x^*)$ defined on $C(\Omega^*)$ satisfying $\|f^*\|=1$. From this follows, by applying the second part of Theorem 1, that there exists a countably additive measure $m^{**}(B^{**})$ defined on the Borel field $\mathfrak{B}^{**}$ satisfying

$$(10) \qquad f^*(x^*)=\int_{\Omega^*} x^*(\omega^*)m^{**}(d\omega^*)$$

for any $x^*(\omega^*)\in C(\Omega^*)$. It is then easy to see that $m^{**}(V^*)$ coincides with (5) if $V^*\in\mathfrak{B}^*\subseteqq\mathfrak{B}^{**}$ is of the form (4), and hence that $m^{**}(B^{**})$ coincides with $m^*(B^*)$ on $\mathfrak{B}^*$. This completes the proof of Theorem 2.

5. A measure space $(\Omega,\mathfrak{B},m)$ is *completed* if $B\in\mathfrak{B}$, $m(B)=0$ and $B'\subseteqq B$ imply $B'\in\mathfrak{B}$ and $m(B')=0$. It is easy to see that for any measure space $(\Omega,\mathfrak{B},m)$, there exists a smallest completed measure space $(\Omega,\overline{\mathfrak{B}},\overline{m})$ which is an extension of $(\Omega,\mathfrak{B},m)$. In fact, $\overline{\mathfrak{B}}$ consists of all subsets $\overline{B}$ of Ω such that there exists two subsets B and N of Ω such that $B\in\mathfrak{B}$, $N\in\mathfrak{B}$, $m(N)=0$ and $\overline{B}\cup B-\overline{B}\cap B\subseteqq N$. This measure space is called the *completion* of $(\Omega,\mathfrak{B},m)$.

It is easy to see that the product measure spaces $(\Omega^*,\mathfrak{B}^*,m^*)$ and $(\Omega^*,\mathfrak{B}^{**},m^{**})$ are not necessarily completed. Let us consider their completions $(\Omega^*,\overline{\mathfrak{B}^*},\overline{m^*})$ and $(\Omega^*,\overline{\mathfrak{B}^{**}},\overline{m^{**}})$. Then

Theorem 3. If each Ω^γ is a compact metric space and if $m^\gamma(O^\gamma)>0$ for any non-empty open subset O^γ of Ω^γ, $\gamma\in\Gamma$, then the two measure spaces $(\Omega^*,\overline{\mathfrak{B}^*},\overline{m^*})$ and $(\Omega^*,\overline{\mathfrak{B}^{**}},\overline{m^{**}})$ coincide. (We do not assume that Γ is countable).

Proof. It suffices to show that every open subset $O^{**}$of Ω^* belongs to $\overline{\mathfrak{B}^*}$. Because of the relation (6), there exists a sequence $\{O_n^*\mid n=1,2,\dots\}$ such that $O_n^*\in\mathfrak{O}^*$, $O_n^*\subseteqq O^{**}$ and $m^*(O_n^*)>m^{**}(O^{**})>1/n$, $n=1,2,\dots$. Then the open set $O^*=\bigcup_{n=1}^\infty O_n^*$ clearly satisfies $O^*\in\mathfrak{O}^*$, $O^*\subseteqq O^{**}$ and $m^{**}(O^{**}-O^*)=0$. Hence it suffices to show that there exists a Borel set $B^*\in\mathfrak{B}^*$ such that $O^{**}-O^*\subseteqq B^*$ and $m^*(B^*)=0$.

Let $\Gamma_0=\{\gamma_n\mid n=1,2,\dots\}\subseteqq\Gamma$ be a countable number of coordinates which determine the set O^*. By means of this system Γ_0 we may decompose Ω^* into two factors:

$$(11) \quad \Omega^*=\Omega^{*(1)}\times\Omega^{*(2)}, \quad \text{where} \quad \Omega^{*(1)}=P_{\gamma\in\Gamma_0}\Omega^\gamma, \quad \Omega^{*(2)}=P_{\gamma\in\Gamma-\Gamma_0}\Omega^\gamma$$

and each $\omega^*\in\Omega^*$ may be expressed in the form: $\omega^*=(\omega^{*(1)},\omega^{*(2)})$, $\omega^{*(1)}\in\Omega^{*(1)}$, $\omega^{*(2)}\in\Omega^{*(2)}$. Let us put $\operatorname{Proj}\omega^*=\omega^{*(1)}$ for any $\omega^*\in\Omega^*$, and $\operatorname{Proj}A^*=\{\operatorname{Proj}\omega^*\mid\omega^*\in A^*\}$ for any subset A^* of Ω^*. Further, let us denote by $m^{*(1)}(B^{*(1)})$ the independent product measure defined on the Borel field $B^{*(1)}=\mathfrak{B}_{\Omega^{*(1)}}$ of all Borel subsets $B^{*(1)}$ of $\Omega^{*(1)}$. Since the space $\Omega^{*(1)}$ is clearly compact metric, and hence separable, we have no need to distinguish the Borel field $\mathfrak{B}^{**(1)}$ and the measure $m^{**(1)}(B^{**(1)})$ from $\mathfrak{B}^{*(1)}$ and $m^{*(1)}(B^{*(1)})$ respectively. It is also clear that

$$(12) \qquad O^*=\operatorname{Proj}O^*\times\Omega^{*(2)}.$$

For each $\omega^* \in O^{**}-O^*$ take a neighborhood $V^*(\omega^*) \in \mathfrak{B}^*$ of ω^* such that $V^*(\omega^*) \subseteqq O^{**}$. Since $V^*(\omega^*) \in \mathfrak{B}^*$, the definition of O^* implies that $m^*\big(V^*(\omega^*) \cup O^*\big) = m^*(O^*)$ or equivalently $m^*\big(V^*(\omega^*) - V^*(\omega^*) \cap O^*\big) = 0$. If we observe that every $V^*(\omega^*)$ is of the form (4), then, by decomposing the finite set $\{\gamma_1, \dots, \gamma_n\}$ into two parts: $\{\alpha_1, \dots, \alpha_p\} \subseteqq \Gamma_0$ and $\{\beta_1, \dots, \beta_q\} \subseteqq \Gamma - \Gamma_0$ we see that

$$(13) \quad \begin{aligned} V^*(\omega^*) &= (O^{\alpha_1} \times \cdots \times O^{\alpha_p} \times P_{\alpha \in \Gamma_0 - \{\alpha_1, \dots, \alpha_p\}} \Omega^\alpha) \\ &\qquad \times (O^{\beta_1} \times \cdots \times O^{\beta_q} \times P_{\beta \in \Gamma - \Gamma_0 - \{\beta_1, \dots, \beta_q\}} \Omega^\beta) \\ &= \text{Proj } V^*(\omega^*) \times O^{\beta_1} \times \cdots \times O^{\beta_q} \times P_{\beta \in \Gamma - \Gamma_0 - \{\beta_1, \dots, \beta_q\}} \Omega^\beta . \end{aligned}$$

Consequently, $V^*(\omega^*) - V^*(\omega^*) \cap O^* = \big(\text{Proj } V^*(\omega^*) - \text{Proj } V^*(\omega^*) \cap \text{Proj } O^*\big) \times O^{\beta_1} \times \cdots \times O^{\beta_q} \times P_{\beta \in \Gamma - \Gamma_0 - \{\beta_1, \dots, \beta_q\}} \Omega^\beta$, and since $m^{\beta_i}(O^{\beta_i}) > 0$, $i = 1, \dots, q$, we finally see that $m^{*(1)}\big(\text{Proj } V^*(\omega^*) - \text{Proj } V^*(\omega^*) \cap \text{Proj } O^*\big) = 0$.

On the other hand, the obvious relation $O^{**} - O^* \subseteqq \bigcup_{\omega^* \in O^{**} - O^*} V^*(\omega^*)$ implies $\text{Proj}\,(O^{**} - O^*) \subseteqq \bigcup_{\omega^* \in O^{**} - O^*} \text{Proj } V^*(\omega^*)$. Since the space $\Omega^{*(1)}$ is compact metric and hence separable, and since each $\text{Proj } V^*(\omega^*)$ is open there exists a countable set $\{\omega_k \mid k = 1, 2, \dots\} \subseteqq O^{**} - O^*$ such that $\text{Proj}\,(O^{**} - O^*) \subseteqq \bigcup_{k=1}^\infty \text{Proj } V^*(\omega_k^*)$, and consequently $\text{Proj}\,(O^{**} - O^*) \subseteqq \bigcup_{k=1}^\infty \big(\text{Proj } V^*(\omega_k^*) - \text{Proj } V^*(\omega_k^*) \cap \text{Proj } O^*\big)$. Thus, by putting $B^* = \bigcup_{k=1}^\infty \big(\text{Proj } V^*(\omega_k^*) - \text{Proj } V^*(\omega_k^*) \cap \text{Proj } O^*\big) \times \Omega^{*(2)}$ we have $B^* \in \mathfrak{B}^*$, $O^{**} - O^* \subseteqq B^*$ and $m^*(B^*) = 0$ as we wanted to prove. This completes the proof of Theorem 3.

6. As an illustration, consider the infinite direct product group $G^* = P_{\gamma \in \Gamma} G^\gamma$, where each G^γ is a group topologically isomorphic with the group of real numbers mod. 1. G is then a compact abelian group which is not metric separable if the set of indices is not countable. The result obtained above shows that it is possible to define two kinds of Haar measure spaces $(G^*, \mathfrak{B}^*, m^*)$ and $(G^*, \mathfrak{B}^{**}, m^{**})$, of which the latter is a proper extension of the former, while these both have the same completion. This follows easily from the fact that we may consider the Haar measure of G^* as an infinite direct product measure of the Haar (=Lebesgue) measures on G^γ, $\gamma \in \Gamma$. An interesting example is given by the case when Γ has exactly the power of continuum. In this case G^* is not separable in the sense of Hausdorff, but there exists a countable subset which is dense in G^*. In fact, we can even show that there exists an element a of the group G^* such that the set $\{a^n \mid n = 1, 2, \dots\}$ is everywhere dense in G^*. This group is, indeed, the only essential example of a non-separable compact abelian group with the said property.