56. *Topological Properties of the Unit Sphere of a Hilbert Space.*

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§1. The well known fixed point theorem of L. E. J. Brouwer concerning continuous mappings of a finite dimensional closed solid sphere into itself was extended by J. Schauder[1] and A. Tychonoff[2] to the case of an infinite dimensional space in the following way:

Let K be a compact (=bicompact) convex set in a locally convex topological linear space X, and let $x' = \varphi(x)$ be a continuous mapping of K into itself. Then there exists a point $x_0 \in K$ such that $x_0 = \varphi(x_0)$.

The purpose of this note is to investigate whether the same or an analogous thing is true for the closed solid unit sphere $K = \{x \mid \|x\| \leqq 1\}$ of a Hilbert space H. Since K is compact with respect to the weak topology[3] of H, the result quoted above implies that there always exists a fixed point for any weakly continuous mapping of K into itself. Concerning strongly continuous mappings of K into itself, however, there seems to be no published result. In the following lines we shall first show that the fixed point theorem does not hold for strongly continuous mappings of K into itself. In fact, we can even show that there exists a homeomorphism (with respect to the strong topology of H) of K onto itself which has no fixed point at all (Theorem 1). This result will then be applied to show that the surface $S = \{x \mid \|x\| = 1\}$ of K is a retract (in the sense of K. Borsuk[4]) in K (Theorem 2), and further that the identity mapping $\varphi_1(x) \equiv x$ is homotopic with the constant mapping $\varphi_0(x) \equiv x_0$ on S (Theorem 3). The paper is concluded with some unsolved problems.

§2. *Theorem 1. There exists a homeomorphism (with respect to the strong topology) $x' = \varphi(x)$ of the closed solid unit sphere $K = \{x \mid \|x\| \leqq 1\}$ of a Hilbert space H onto itself which has no fixed point.*

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1) J. Schauder, Der Fixpunktsatz im Funktionalräumen, Studia Math. **2** (1930), 171–180.

2) A. Tychonoff, Ein Fixpunktsatz, Math. An. **111** (1935), 767–776.

3) Cf. S. Kakutani, Weak topology and regularity of Banach spaces, Proc. **15** (1939), 169–173, and S. Kakutani, Weak topology, bicompact set and the principle of duality, Proc. **16** (1940), 63–67. The weak topology of a Banach space X is defined as follows: For any $x_0 \in X$, its weak neighborhood $V(x_0)$ is defined by $V(x_0) = \{x \mid |f_i(x) - f_i(x_0)| < \varepsilon,\ i = 1, \dots, n\}$, where $\{f_1, \dots, f_n\}$ is an arbitrary finite system of bounded linear functionals on X and $\varepsilon > 0$ is an arbitrary positive number. It is known (Theorem 3 of the second paper of the author quoted above) that a necessary and sufficient condition for a Banach space X to be regular (=reflexive) is that the closed solid unit sphere $K = \{x \mid \|x\| \leqq 1\}$ of X is compact with respect to the weak topology of X. Since a Hilbert space H is reflexive, the closed solid unit sphere K of H is compact with respect to the weak topology of H.

4) K. Borsuk, Sur les rétractes, Fund. Math. **17** (1931), 152–170. A subset B of a topological space A is a retract in A if there exists a continuous mapping $x' = \varphi(x)$ of A onto B such that $\varphi(x) \equiv x$ on B.

Proof. Let $\{y_n \mid n=0, \pm 1, \pm 2, \dots\}$ be a complete orthogonal normalized system of H, and let $x'=U(x)$ be a unitary transformation of H onto itself which is defined by the relations: $y_{n+1}=U(y_n)$, $n=0, \pm 1, \pm 2, \dots$. Let us then put

$$\varphi(x)=\frac{1}{2}(1-\|x\|)y_0+U(x)\,. \tag{1}$$

We shall show that $x'=\varphi(x)$ is a mapping with a required property.

In order to show that $x'=\varphi(x)$ is a homeomorphism of K onto itself, we put $\varphi(x)$ into the form:

$$\varphi(x)=(1-\|x\|)\cdot\frac{y_0}{2}+\|x\|\cdot U(y)\,, \qquad x \neq 0\,, \tag{2}$$

where $y=x/\|x\|$ is a point in which (the extension of) the vector $\overrightarrow{0x}$ (where 0 is the origin of H) meets the surface $S=\{x \mid \|x\|=1\}$ of K. Then (2) shows that $\varphi(x)$ is a point which divides the segment $\overline{y_0/2, U(y)}$ in the same proportion as the point x divides the segment $\overline{0, y}$. Since $x'=U(x)$ is a homeomorphism of S onto itself, it follows easily that $x'=\varphi(x)$ is a homeomorphism of K onto itself.

Let us now assume that there exists a point $x_0 \in K$ such that $x_0=\varphi(x_0)$. Then we must have

$$x_0-U(x_0)=\frac{1}{2}(1-\|x_0\|)y_0\,. \tag{3}$$

From this follows that $x_0 \neq 0$. Further, since $x'=U(x)$ is a homeomorphism of S onto itself without fixed point on S, we must have $\|x_0\|<1$. Let us consider the expansion of x:

$$x_0=\textstyle\sum_{n=-\infty}^{\infty}a_n y_n\,, \quad \text{where} \quad \sum_{n=-\infty}^{\infty}|a_n|^2=\|x_0\|^2<1\,. \tag{4}$$

Then $x_0-U(x_0)=\sum_{n=-\infty}^{\infty}a_n y_n-\sum_{n=-\infty}^{\infty}a_n y_{n+1}=\sum_{n=-\infty}^{\infty}(a_n-a_{n-1})y_n$, which together with (3) imply $a_0-a_{-1}=\frac{1}{2}(1-\|x_0\|)>0$ and $a_n-a_{n-1}=0$, $n=\pm 1, \pm 2, \dots$, and hence $\dots=a_{-3}=a_{-2}=a_{-1}<a_0=a_1=a_2=a_3=\cdots$. Since this is clearly incompatible with the fact that $\sum_{n=-\infty}^{\infty}|a_n|^2<\infty$, the mapping $x'=\varphi(x)$ cannot have any fixed point in K.

Theorem 2. The surface $S=\{x \mid \|x\|=1\}$ of K is a retract in K, i.e., *there exists a continuous mapping $x'=\psi(x)$ of K onto S such that $\psi(x)\equiv x$ on S.*

Proof. Let $\psi(x)$ be a point in which (the extension of) the vector $\overrightarrow{\varphi(x), x}$ meets the surface S of K, where $x'=\varphi(x)$ is a homeomorphism of K onto itself (having no fixed point) obtained in Theorem 1. Then it is easy to see that $x'=\psi(x)$ is a continuous mapping of K onto S such that $\psi(x)\equiv x$ on S.

Theorem 3. The identity mapping $\varphi_1(x)\equiv x$ is homotopic with the constant mapping $\varphi_0(x)\equiv x_0$ on S, i.e., *there exists a continuous map-*

ping $x'=\psi(x,t)$ *of* $S\times(0,1)$[1] *onto* S *such that* $\psi(x,0)\equiv x_0$ *on* S *and* $\psi(x,1)\equiv x$ *on* S.

Proof. It suffices to put $\psi(x,t)=\psi(tx)$, where $x'=\psi(x)$ is a continuous mapping of K onto S obtained in Theorem 2.

§ 3. *Unsolved problems.* Is K homeomorphic with S? Is K or S homeomorphic with H? Is there any homeomorphism of K or H onto itself of finite period without fixed point? If $x'=\varphi(x)$ is a homeomorphism of K onto itself, is it necessary that the surface S of K is mapped onto itself by $x'=\varphi(x)$? How is the situation in general Banach spaces?

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1) $S\times(0,1)$ means the set of all pairs (x,t), where $x\in S$ and $0\leqq t\leqq 1$, with the usual product topology of the Cartesian product.