PAPERS COMMUNICATED

52. *On the Riemann Surface of an Inverse Function of a Meromorphic Function in the Neighbourhood of a Closed Set of Capacity Zero.*

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Let E be a closed set of capacity zero[1] on the z-plane and C be a Jordan curve surrounding E and D be the domain bounded by E and C. Let $w=w(z)$ be one-valued and meromorphic in D and on C and have an essential singularity at every point of E and F be the Riemann surface of the inverse function $z=z(w)$ of $w=w(z)$ spread over the w-plane. Concerning F, the following facts are known: (i) F covers any point on the w-plane infinitely many times, except a set of points of capacity zero[2]. (ii) Let w_0 be a regular point of F. Then $z(w)$ can be continued analytically on the half-lines: $w=w_0+re^{i\theta}$ $(0\leqq r<\infty)$ indefinitely or till we meet the image of C, except a set of values of θ of measure zero[3].

Let (w_0) be a boundary point of F, whose projection on the w-plane is w_0. Iversen called (w_0) a direct transcendental singularity of $z(w)$, if w_0 is lacunary for a connected piece F_0 of F, which lies above a disc K_0 about w_0 and has (w_0) as its boundary point.

We will prove the following third property of F.

Theorem. The set of points on the w-plane, which are the projections of direct transcendental singularities of $z(w)$ is of capacity zero.

We will first prove a lemma.

Lemma. Let F_0 be a connected piece of a Riemann surface F spread over the w-plane, which lies above a disc K_0 bounded by a circle C_0. Suppose that F_0 does not cover a closed set E_0, which lies with its boundary inside C_0. If there exists a non-constant $f(w)$ on F_0, which satisfies the following conditions: (i) *$f(w)$ is one-valued and meromorphic on F_0,* (ii) *$f(w)$ does not take the values on a closed set E of capacity zero,* (iii) *$f(w)$ tends to E, when w tends to any accessible boundary point of F_0, whose projection lies inside C_0, then* $\text{cap.}\,E_0=0$.

Proof. Let $\mathfrak{F}$ be the simply connected universal covering Riemann surface of F_0. We map $\mathfrak{F}$ on $|x|<1$ by $w=\varphi(x)$. Suppose that $\text{cap.}\,E_0>0$, then, as I have proved in my former paper[4], the accessible

1) In this note, "capacity" means "logarithmic capacity."

2) R. Nevanlinna: Eindeutige analytische Funktionen. p. 132. Satz 2. S. Kametani: The exceptional values of functions with the set of capacity zero of essential singularities. Proc. **17** (1941).

3) M. Tsuji: On the behaviour of a meromorphic function in the neighbourhood of a closed set of capacity zero. Proc. **18** (1942).

4) M. Tsuji: On the domain of existence of an implicit function defined by an integral relation $G(x,y)=0$. Proc. **19** (1943).

boundary points of F_0, whose projections lie inside C_0, correspond to a set e_0 of positive measure on $|x|=1$. Since cap. $E=0$, by Evans' theorem[5], there exists a positive mass-distribution $d\mu(a)$ of total mass 1 on E, such that

$$u(z)=\int_E \log\frac{1}{|z-a|}d\mu(a)$$

tends to $+\infty$, when z tends to any point of E. Let $v(z)$ be the conjugate harmonic function of $u(z)$ and put $H(z)=e^{-(u+iv)}$. Then $H(z)$ is meromorphic outside E and tends to zero, when z tends to any point of E. We put $G(x)=H\Big(f\big(\varphi(x)\big)\Big)$. Then $G(x)$ is one-valued and meromorphic in $|x|<1$.

If x tends to any point of e_0 non-tangentially to $|x|=1$, $f\big(\varphi(x)\big)$ tends to E, so that $G(x)$ tends to zero. Hence by Priwaloff's theorem, $G(x)\equiv 0$, or $f(w)\equiv$ const., which contradicts the hypothesis. Hence cap. $E_0=0$, q. e. d.

By this lemma, we can prove the Theorem as follows.

Let F_0 be a connected piece of F, which lies above a disc K_0 bounded by a circle C_0. We suppose that F_0 does not contain the image of C.

We see easily that, $z(w)$ tends to E, when w tends to any accessible boundary point of F_0, whose projection lies inside C_0. Since $z(w)$ does not take the values on E, we have by the Lemma, that the set of points inside C_0, which are uncovered by F_0 is of capacity zero. This point established, we can proceed similarly as in my former paper[6] and prove the Theorem.

5) Evans: Potentials and positively infinite singularities of harmonic functions. Monathefte f. Math. u. Phys. **43** (1936).

6) M. Tsuji, l. c. 4).