94. *Normed Rings and Spectral Theorems, II.*

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§ 1. *Introduction.* The purpose of this note is to give an algebraic treatment and version of Fredholm-Riesz-Schauder's theory[1] of completely continuous (c.c.) functional equations with the aide of the theory of normed ring[2]. Our method seems to be suited to obtain the results concerning conjugate equations. We also give a proof to S. Nikolski's extension[3], which is of importance in view of applications, of F-R-S's theory. Lastly we extend the existence theorem of proper values $\neq 0$ for c.c. hermitian operator $\neq 0$ as a corollary of our arguments.

§ 2. *Preliminaries and lemmas.* Let V be a linear operator from a complex Banach space E into E. V is called c.c. if it transforms any bounded set into a compact set. Let $\boldsymbol{R}$ be the commutative ring generated by the c.c. V and the identity operator I, completed by the uniform limit defined by the norm $\|T\| = \sup_{|x| \leq 1} \|T \cdot x\| \cdot \boldsymbol{R}$ is a normed ring with unit I and the norm $\|T\|$, such that any element $T \in \boldsymbol{R}$ may be represented as $T = \lambda I - U$, U being c.c. Let E^* denote the conjugate space $\left(= \text{the space of all the linear functionals } f \text{ on } E \text{ with the norm } \|f\| = \sup_{|x| \leq 1} |f(x)|\right)$, then the operator T^* conjugate to T is defined by $T^* \cdot f = g$, $f(T \cdot x) = g(x)$, $f \in E^*$.

Lemma 1. Since[4] $\|T\| = \|T^*\|$, the set $\boldsymbol{R}^*$ of all the operators T^*, $T \in \boldsymbol{R}$, is also a normed ring linear-isomorphic and linear-isometric with $\boldsymbol{R}$ by the correspondence $T \leftrightarrow T^*$.

Lemma 2[5]. $\boldsymbol{R}$ is, as a normed ring, linear homomorphically mapped upon a complex-valued function ring defined on the space $\mathfrak{M}$ of all the maximal ideals $\boldsymbol{M}$ of $\boldsymbol{R}$: $\boldsymbol{R} \ni T \to T(\boldsymbol{M})$, $I \to I(\boldsymbol{M}) \equiv 1$ such that $T \equiv T(\boldsymbol{M})I \pmod{\boldsymbol{M}}$, $\sup_{\boldsymbol{M}} |T(\boldsymbol{M})| = \lim_{n \to \infty} \sqrt[n]{\|T^n\|}$. Moreover T admits inverse T^{-1} in $\boldsymbol{R}$ if and only if $T(\boldsymbol{M}) \neq 0$ on $\mathfrak{M}$.

Lemma 3[6]. Let $I_1 \in \boldsymbol{R}$ be an idempotent viz. $I_1^2 = I_1$, then the

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1) See, for example, S. Banach's book: Théorie des opérations linéaires, Warsaw (1932), 151. Cf. M. Nagumo: Jap. J. of Math., **13** (1936), 6.

2) I. Gelfand: Rec. Math., **9** (1941), 3.

3) C. R. URSS, **16** (1926), 315. Cf. also K. Yosida: Jap. J. Math., **15** (1939), 297.

4) S. Banach: loc. cit., 100.

5) I. Gelfand: loc. cit.

6) Cf. I. Gelfand: loc. cit., 18. For the sake of completeness we will give the proof below. For any $T \in \boldsymbol{R}$ we have $(T - T(\boldsymbol{M})I)I_1 \in \boldsymbol{M} \frown \boldsymbol{R}_1$, since $(T - T(\boldsymbol{M})I) I_1(\boldsymbol{M}) = (T(\boldsymbol{M}) - T(\boldsymbol{M})1)(I_1(\boldsymbol{M})) = 0$, $(T - T(\boldsymbol{M})I)I_1 \cdot I_1 = (T - T(\boldsymbol{M})I)I_1$. Thus $TI_1 = T(\boldsymbol{M})I_1 + (T - T(\boldsymbol{M})I)I_1 \equiv T(\boldsymbol{M})I_1 \pmod{\boldsymbol{M} \frown \boldsymbol{R}_1}$, proving that $\boldsymbol{M}' = \boldsymbol{M} \frown \boldsymbol{R}_1$ is a maximal ideal of $\boldsymbol{R}_1$. Next let $\boldsymbol{M}'$ be a maximal ideal of $\boldsymbol{R}_1$. We will show that there exists a maximal ideal $\boldsymbol{M}$ of $\boldsymbol{R}$ such that $\boldsymbol{M} \geqq \boldsymbol{M}'$, $\boldsymbol{M} \bar{\ni} I_1$. To this purpose consider $\boldsymbol{M} = \boldsymbol{M}' + \boldsymbol{R}(I - I_1)$. That $\boldsymbol{M} \bar{\ni} I_1$ is trivial. Since $T = TI_1 + T(I - I_1)$, $TI_1 \equiv \lambda I_1 \pmod{\boldsymbol{M}'}$ by the lemma 2), we obtain $T \equiv \lambda I_1 \pmod{\boldsymbol{M}}$, proving that $\boldsymbol{M}$ is a maximal ideal of $\boldsymbol{R}$.

subring $R_1 = RI_1$ of R constitutes normed ring with unit I_1. For any maximal ideal $M \ni I_1$ of R, $M' = M \frown R_1$ is a maximal ideal of R_1. Conversely for any maximal ideal M' of R_1, there exists a maximal ideal $M \ni I_1$ of R such that $M' = M \frown R_1$.

Lemma 4[1]. Let E_1 be a closed linear proper subspace of E, then there exists $x_0 \in E - E_1$ such that $\|x_0\| = 1$, $\text{dis}(x_0, E_1) = \inf_{x \in E_1} \|x_0 - x\| = \frac{1}{2}$.

§ 3. *A deduction of F-R-S's theory.* By the lemma 2

(1) *the range $R(V)$ of the function $V(M)$ coincides with the spectra of V,*

viz. with the set of complex numbers λ such that $T = \lambda I - V$ does not have inverse T^{-1} in R. Thus, by the isomorphism $R \leftrightarrow R^*$,

(2) *the spectra are the same for V and for V^*.*

We first show that

(3) *any complex number $\neq 0$ from the spectra of V is a proper value of V.*

Proof. Let $\lambda = 1 \in R(V)$ and let $\lambda = 1$ be not a proper value of V. Then $T = I - V$ maps E on $T \cdot E$ in one-to-one manner. The inverse mapping T^{-1} is continuous from $T \cdot E$ on E viz. there exists a positive number $\alpha > 0$ such that $\|T \cdot x\| \geqq \alpha \|x\|$ on E. Assume the contrary and let $\lim_{n\to\infty} \|T \cdot x_n\| = 0$, $\|x_n\| = 1$ $(n = 1, 2, \dots)$. By the c.c. of V we may suppose that $\lim_{n\to\infty} V \cdot x_n = y$ exists, whence $\lim_{n\to\infty} (x_n - V \cdot x_n) = 0$ or $\lim_{n\to\infty} x_n = y$, $\|y\| = 1$, $y = V \cdot y$, contrary to the hypothesis. Thus T^{-1} must be continuous from $T \cdot E$ on E and hence $T \cdot E$ is a closed set of E. $T \cdot E \neq E$, since otherwise T^{-1} exists in R and thus $\lambda = 1 \bar{\in} R(V)$. Hence, if we put $E_1 = T \cdot E$, $E_2 = T \cdot E_1, \dots, E_{n+1}$ is a closed linear proper subspace of E_n $(n = 1, 2, \dots)$. By the lemma 4, there exists a sequence $\{y_n\}$ such that $y_n \in E_n$, $\|y_n\| = 1$, $\text{dis}(y_n, E_{n+1}) = \frac{1}{2}$. Hence, for $n > m$, $V \cdot y_m - V \cdot y_n = y_m - (y_n + T \cdot y_m - T \cdot y_n) = y_m - y$, $y \in E_{m+1}$ and therefore $\|V \cdot y_m - V \cdot y_n\| \geqq \frac{1}{2}$, contrary to the c.c. of V. Q. E. D

Remark 1. Let E be a general euclid space and let V be a c.c. *normal* operator in E. Since[2], by the normality, $\|T\| = \lim_{n\to\infty} \sqrt[n]{\|T^n\|}$, $T = \lambda I - U$, the mapping $T \to T(M)$ is isomorphic. Thus, if $V \neq 0$, $R(V)$ contains complex numbers $\neq 0$. Hence V admits proper values $\neq 0$, in accordance with the well-known theorem due to Hilbert-Schmidt[3].

Since V is c.c. we have, by (3),

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1) S. Banach: loc. oit., 83.

2) See K. Yosida: Proc. **19** (1943), 338.

3) A short cut to *H-S*'s theorem is given by Y. Mimura: 全國紙上數學談話會, 第 170 號 (昭和 14 年).

(4) *the spectra of V constitute an enumerable set which may accumulate only at* 0[1].

Next let $\lambda=1$ be a proper value of V. *We will show that* $\lambda=1$ *is also a proper value of* V^*, without making use of the fact that V^* is c.c. with V.

Proof. By (1) and (4) there exists $\varepsilon>0$ such that $(\lambda I-V)^{-1}$ exists in $\boldsymbol{R}$ if $0<|\lambda-1|<3\varepsilon$. Consider the resolvent integral

$$(5) \qquad I_1=\frac{1}{2\pi i}\int_{|\lambda-1|=\delta}(\lambda I-V)^{-1}d\lambda\,, \qquad 0<\delta<3\varepsilon\,.$$

By Cauehy's theorem, I_1 is independent of δ. Thus

$$(6) \qquad I_1^2=\frac{1}{2\pi i}\int_{|\lambda-1|=\varepsilon}(\lambda I-V)^{-1}d\lambda\cdot\frac{1}{2\pi i}\int_{|\mu-1|=2\varepsilon}(\mu I-V)^{-1}d\mu$$

$$=\frac{1}{2\pi i}\int_{|\lambda-1|=\varepsilon}\left(\frac{1}{2\pi i}\int_{|\mu-1|=2\varepsilon}\frac{d\mu}{\mu-\lambda}\right)(\lambda I-V)^{-1}d\lambda$$

$$-\frac{1}{2\pi i}\int_{|\mu-1|=2\varepsilon}\left(\frac{1}{2\pi i}\int_{|\lambda-1|=\varepsilon}\frac{d\lambda}{\mu-\lambda}\right)(\mu I-V)^{-1}d\mu$$

$$=\frac{1}{2\pi i}\int_{|\lambda-1|=\varepsilon}(\lambda I-V)^{-1}d\lambda=I_1\,.$$

Hence we obtain the direct decomposition

$$(7) \qquad \boldsymbol{R}=\boldsymbol{R}I_1+\boldsymbol{R}(I-I_1)\,.$$

By Cauchy's theorem and the lemma 2 we have

$$\begin{cases} I_1(\boldsymbol{M}_1)=\dfrac{1}{2\pi i}\displaystyle\int_{|\lambda-1|=\varepsilon}(\lambda\cdot1-1)^{-1}d\lambda=1 & \text{if} \quad V(\boldsymbol{M}_1)=1\,, \\ I_1(\boldsymbol{M}_\mu)=\dfrac{1}{2\pi i}\displaystyle\int_{|\lambda-1|=\varepsilon}(\lambda\cdot1-\mu)^{-1}d\lambda=0 & \text{if} \quad V(\boldsymbol{M}_\mu)=\mu\neq1\,. \end{cases}$$

Hence, by the lemma 3, we obtain the results:

$$\begin{cases} \boldsymbol{M}_1'=\boldsymbol{M}_1\frown\boldsymbol{R}I_1 \ \text{ is the only maximal ideal of } \boldsymbol{R}I_1, \\ \boldsymbol{M}_\mu'=\boldsymbol{M}_\mu\frown\boldsymbol{R}(I-I_1) \ \text{ with } \mu\neq1 \text{ exaust the maximal ideals of } \boldsymbol{R}(I-I_1)\,. \end{cases}$$

Therefore, siece $(I-V)(I-I_1)(\boldsymbol{M}_\mu)_{\mu\neq1}=(1-\mu)(1-0)\neq0$,

(8) $(I-V)(I-I_1)$ *admits inverse* $\big($*in* $\boldsymbol{R}(I-I_1)\big)$ *as an element of the ring* $\boldsymbol{R}(I-I_1)$.

Moreover, since $(I-V)I_1(\boldsymbol{M}_1)=(1-1)1=0$,

(9) $(I-V)I_1$ is, as an element of $\boldsymbol{R}I_1$, contained in all the (in the truth, only one) maximal ideals of $\boldsymbol{R}I_1$.

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1) S. Banach: loc. cit.. 160.

On the other hand, if we put $(\lambda I-V)^{-1}=\frac{I}{\lambda}-R_\lambda$ then $R_\lambda=\frac{R_\lambda}{\lambda}\cdot V-\frac{V}{\lambda^2}$ is c.c. with V. Thus, by Cauchy's theorem,

(10) $I_1=\frac{1}{2\pi i}\int_{|\lambda-1|=\epsilon}\left(\frac{I}{\lambda}-R_\lambda\right)d\lambda=\frac{-1}{2\pi i}\int_{|\lambda-1|=\epsilon}R_\lambda d\lambda$ *is c.c.*[1]

A Banach space is, by the lemma 4, of finite dimension if and only if its unit sphere is compact[2]. Hence, by the c.c. and the idempotent character of I_1, $I_1\cdot E$ is of finite dimension. Therefore

(11) *the ring $RI_1=I_1RI_1$ may be considered as a commutative matrix ring with unit I_1 operating upon the vector space $I_1\cdot E$ of finite dimension.*

Thus, by (9), the matrix $(I-V)I_1$ must be nilpotent. Hence

(12) *$(I-V)I_1$ is nilpotent.*

$I_1\neq 0$, since otherwise $T=I-V$ admits inverse in $\boldsymbol{R}$ by (8), contrary to the hypothesis $\lambda=1\in R(V)$. By the isomorphism $\boldsymbol{R}\leftrightarrow\boldsymbol{R}^*$, (5)–(12) hold good when we assign *-notation to every operator or set in these equations ($\boldsymbol{M}^*$, for example, means the set $\mathop{\mathcal{E}}_{T^*}\{T\in\boldsymbol{M}\}$). Thus, by the formula corresponding to (12), we see that $\lambda=1$ is also a proper value of V^*. (We will denote the formulae corresponding to (5)-(12) by (5)*–(12)*). Q. E. D.

Now let $\lambda=1$ be a proper value of V and of V^*. Then

(13) *the dimension of the proper space belonging to the proper value 1 is the same for V and for V^*.*

Proof. By (7) and (8) the proper value equation

(14) $$x=V\cdot x, \qquad x\in E$$

is equivalent to $(I-V)I_1\cdot x=0$, $(I-I_1)\cdot x=0$. Thus (14) is equivalent to

(14)′ $$x=V\cdot x, \qquad x\in I_1\cdot E.$$

Similarly, by (7)* and (8)*, the proper value equation

(15) $$f=V^*\cdot f, \qquad f\in E^*$$

is equivalent to

(15)′ $$f=V^*\cdot f, \qquad f\in I_1^*\cdot E^*$$

Since $I_1^*\cdot f=f$ means $f\big((I-I_1)\cdot x\big)=0$ on E, $I_1^*\cdot E$ must be of the same dimension as $I_1\cdot E$. Therefore the mutual conjugate matrix equation

1) A uniform limit of a sequence of c.c. linear operators is c.c. also. See S Banach: loc. cit., 96.

2) S. Banach: loc. cit., 84.

(14)′ and (15)′ admit respectively the same number of linearly independent solutions. Q. E. D.

Lastly we will prove that, if $\lambda=1$ is a proper value of V and of V^*, then

(16) *the equation* $y=(I-V)\cdot x$ *admits solution* x *if and only if* $f(y)=0$ *when* $f=V^*\cdot f$,

and similarly

(16)′ *the equation* $g=(I^*-V^*)\cdot f$ *admits solution* f *if and only if* $g(x)=0$ *when* $x=V\cdot x$.

Proof of (16). The necessity is trivial. Because of (8) and (11), $(I-V)\cdot E$ is a closed set of E. Thus, if $y\bar{\in}(I-V)\cdot E$, there exists by Hahn-Banach's theorem $f\in E^*$ such that $f(y)\neq 0$, $f\big((I-V)\cdot E\big)=0$, contrary to the hypothesis. Q. E. D.

§ 4. *An extension.* *F-R-S*'s theory may be extended to linear operator V which satisfies

(17) V^n *is c.c. for a certain* $n\geqq 1$.

We will show that this extension, first pointed out by S. Nikolski, is obtained as a corollary to our arguments in § 3.

Since, by the lemma 2, the spectra of V are contained in $\{\lambda^{\frac{1}{n}}\}$, $\lambda\in R(V^n)$, (4) holds goods for our V. Moreover if $\lambda=1\in R(V)$, then I_1 is, as will be proved below, c.c. These two facts would be sufficient for the validity of our extension, as will be verified reflecting upon the arguments in § 3.

Proof of the c.c. of I_1. By the assumption $\lambda=1\in R(V^n)$. Let $(\lambda^n I-V^n)=\frac{I}{\lambda^n}-R_\lambda(n)$, then $R_\lambda(n)$ is c.c. with V. From

$$I=(\lambda I-V)\Big(\frac{I}{\lambda}-R_\lambda\Big),$$

$$I=(\lambda^n I-V^n)\Big(\frac{I}{\lambda^n}-R_\lambda(n)\Big)=(\lambda I-V)(\lambda^{n-1}I+\lambda^{n-2}V+\cdots+V^{n-1})\Big(\frac{I}{\lambda^n}-R_\lambda(n)\Big)$$

we obtain

$$\frac{I}{\lambda}-R_\lambda=\frac{I}{\lambda}+\frac{V}{\lambda^2}+\cdots+\frac{V^{n-1}}{\lambda^n}-R_\lambda(n)(\lambda^{n-1}I+\lambda^{n-2}V+\cdots+V^{n-1}).$$

Thus, by Cauchy's theorem and the c.c. of $R_\lambda(n)$,

$$I_1=\frac{1}{2\pi i}\int_{|\lambda-1|=\epsilon}\Big(\frac{I}{\lambda}-R_\lambda\Big)d=c.c. \qquad \text{Q. E. D.}$$

Remark 2. The proposition stated in the remark 1 is valid for normal operators satisfying (17).