109. Bohr Compactifications of a Locally Compact Abelian Group II.

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(Comm. by T. TAKAGI, Nov. 12, 1943.)

This is a continuation of our preceding paper with the same title.[1] As an application of the general theory developed in the first paper, we shall here discuss monothetic and solenoidal groups.

§ 5. *Monothetic groups.* A topological group G is *monothetic*,[2] if there exists an element $a \in G$, called a *generating element* of G, such that the cyclic subgroup $H=\{a^n \mid n=0, \pm 1, \pm 2, \ldots\}$ of H generated by a is everywhere dense in G. A monothetic group is obviously abelian.

Theorem 5.[3] *A locally compact monothetic group G is either compact or topologically isomorphic with the additive group of all integers with discrete topology.*

Proof. Let us apply Theorem 1, taking as H the additive group of all integers with discrete topology. Then there exists a continuous isomorphism $a^{*\prime}=\varphi^*(a^*)$ of the character group G^* of G onto a topological subgroup $G^{*\prime}$ of the character group H^* of H which is nothing but the additive group K of all real numbers mod. 1 with the usual compact topology. It suffices to show that G^* is either compact or discrete. If G^* is not totally disconnected, then the image $\varphi^*(V^*)$ in K of every open neighborhood V^* of the zero element of G^* contains a continuum and hence a certain interval of K containing the zero element of K. From this follows that $a^{*\prime}=\varphi^*(a^*)$ is an open mapping and so is a homeomorphism. Thus, if G^* is not totally disconnected, then G^* must be topologically isomorphic with K. This, however, happens, only if G is discrete and is topolо́gically isomorphic with H itself. On the other hand, if G^* is totally disconnected, then there exists an arbitrary small subgroup of G^* which is open-and-closed. But this is possible only if G^* is discrete; for, as a topological subgroup of K, the continuous image $G^{*\prime}$ of G^* has no sufficiently small subgroup except the trivial one consisting only of the zero element. Thus G^* must be discrete in this case, and so $G=G^{**}$ must be compact. This completes the proof of Theorem 5.

Remark 5. It is not difficult to construct an example of a complete metric monothetic group which is not locally compact. In fact, let $f^*(n)$ be a complex-valued bounded function defined on the additive group $H=\{n \mid n=0, \pm 1, \pm 2, \ldots\}$ of all integers with the following properties: (i) there exists a sequence $\{m_k \mid k=1, 2, \ldots\}$ of positive

1) H. Anzai and S. Kakutani, Proc. **19** (1943), 476–480.
2) Cf. D. van Dantzig, Compositio Math. **3** (1936), 408–426.
3) This theorem is due to A. Weil, L'intégration dans les groupes et leurs applications, Actualités, 1939. The proof given in this paper is new.

integers such that

(11) $$\lim_{k\to\infty} \sup_n |f^*(n)-f^*(n+m_k)|=0\,,$$

(ii) $f^*(n)$ is not a Bohr almost periodic function on H. If we now put

(12) $$d(n, p)=\sup_k |f^*(n+m_k)-f^*(p+m_k)|\,,$$

then H becomes a metric group with respect to this metric (12), and it is easy to see that the completion of H gives an example of a group with the required properties.

From Theorem 2 follows immediately:

Theorem 6.[1] *In order that a compact abelian group G be monothetic, it is necessary and sufficient that the discrete character group G^* of G be algebraically isomorphic with an algebraic subgroup of the additive group $RE1=K^{(d)}$ of all real numbers mod. 1 with discrete topology.*

Further, from Theorem 3 follows:

Theorem 7. Every compact monothetic group G can be obtained in the following way: take the additive group $H=\{n \mid n=0, \pm 1, \pm 2, \dots\}$ of all integers with discrete topology, and introduce on it a uniform structure (as in Theorem 3) by means of a family $F^*=\{f^*(n)\}$ of complex-valued almost periodic functions $f^*(n)$ defined on H, or by means of a family $X^*=\{a_\lambda^*\}$ of continuous characters a_λ^* on H, where $0\leqq\lambda<1$ and $(n, a_\lambda^*)=\lambda n$ (mod. 1). Then G is obtained by completing H with respect to this uniform structure. G is separable if and only if $X^*(F^*)=\bigcup_{f^*\in F^*}X^*(f^*)$ or X^* is a countable set, where $X^*(f^*)$ is a countable set of characters for which the Fourier coefficient of $f^*(n)$ does not vanish; and so there exists a compact monothetic group which is not separable.

Let us now consider a compact monothetic group G which is totally disconnected. Then every element of the discrete character group G^* of G is of finite order. Hence, from Theorem 6 follows that G^* is algebraically isomorphic with an algebraic subgroup $G^{*\prime}$ of the additive group $RA1$ of all rational numbers mod. 1. Further, it is easy to see that there exists a sequence $\{n_k \mid k=1, 2, \dots\}$ of positive integers such that G^* is algebraically isomorphic with the additive group $RA1(n_1, n_2, \dots)$ of all rational numbers r (mod. 1) of the form $r=m/n_1\cdots n_k$, $m=0, \pm 1, \pm 2, \cdots$ and $k=1, 2, \cdots$ Since $G^*=RA1(n_1, n_2, \dots)$ is clearly the union of an increasing sequence $\{G_k^* \mid k=1, 2, \dots\}$ of finite cyclic subgroups $G_k^*=RA1(n_1, \dots, n_k, 1, 1, \dots)$ of order $n_1\cdots n_k$, so we see

Theorem 8. Every totally disconnected compact monothetic group G can be obtained as a limit group of a sequence $\{C_{n_1,\dots,n_k} \mid k=1, 2, \dots\}$ of finite cyclic groups $C_{n_1,\dots,n_k}$ of order $n_1\cdots n_k$, where $\{n_k \mid k=1, 2, \dots\}$ is a sequence of positive integers and G is finite if and only if there exists a k_0 such that $n_k=1$ for all $k\geqq k_0$.

1) Cf. P. R. Halmos and H. Samelson, Proc. N.A.S. **28** (1942).

This limit group is denoted by $C(n_1, n_2, \ldots)$. Thus every totally disconnected compact monothetic group is separable. It is further to be noted that there is essentially only one homomorphism of $C_{n_1,\ldots,n_k,n_{k+1}}$ onto $C_{n_1,\ldots,n_k}$. From this follows that any two generating elements of $C(n_1, n_2, \ldots)$ are conjugate with each other, i. e. for any two generating elements a and a' of $C(n_1, n_2, \ldots)$, there exists a homeomorphic automorphism of $C(n_1, n_2, \ldots)$ which carries a onto a over. Finally, $C(n_1, n_2, \ldots)$ is a continuous homomorphic image of $C(m_1, m_2, \ldots)$ if and only if, for any k, there exists an l such that $n_1 \cdots n_k$ divides $m_1 \cdots m_l$. Thus $n_k = k$, $k=1, 2, \ldots$, gives a totally disconnected compact monothetic group $G = C(1, 2, \ldots)$ such that every totally disconnected compact monothetic group is a continuous homomorphic image of G.

This result was partly obtained by D. van Dentzig.[1] This limit group $C(1, 2, \ldots)$ is nothing but the compact character group of the additive group $RA1 = RA1(1, 2, \ldots)$ of all rational numbers mod. 1 with discrete topology, and is called the *universal totally disconnected compact monothetic group* or simply the *universal monothetic Cantor group.*

In order to discuss a general case we need

Lemma 1. Let G^* be a discrete abelian group whose cardinal number $\mathfrak{p}(G^*)$ does not exceed $\mathfrak{c}$. Then, in order that G^* be algebraically isomorphic with an algebraic subgroup of the additive group $RE1 = K^{(d)}$ of all real numbers mod. 1 with discrete topology, it is necessary and sufficient that the subgroup G_0^* of G^* consisting of all elements of G^* of finite order be algebraically isomorphic with an algebraic subgroup of the additive group $RA1$ of all rational numbers mod. 1.

The condition of Lemma 1 is clearly necessary. That it is also sufficient may be proved by constructing a required isomorphism by transfinite induction.

Theorem 9. In order that a compact abelian group G be monothetic, it is necessary and sufficient that the cardinal number $\mathfrak{p}(G)$ of G do not exceed $2^{\mathfrak{c}}$ and further that the totally disconnected factor group G/N of G by the component N of the zero element of G be monothetic. In particular, every connected compact abelian group is monothetic whenever $\mathfrak{p}(G) \leqq 2^{\mathfrak{c}}$.

Theorem 9 follows from Theorem 5 and Lemma 1 if we observe the following two facts: (i)[2] $\mathfrak{p}(G^*) \leqq \mathfrak{c}$ is equivalent with $\mathfrak{p}(G) \leqq 2^{\mathfrak{c}}$, (ii)[3] G_0^* is algebraically isomorphic with the discrete character group $(G/N)^*$ of the factor group G/N. In case G satisfies the 2nd countability axiom of Hausdorff, this last statement of Theorem 9 was obtained by J. Schreier and S. M. Ulam[4] by a different method. We may also prove

Theorem 10. A compact abelian group G is monothetic if and only

1) D. van Dantzig, loc. cit. 2).
2) S. Kakutani, On cardinal numbers related with a compact abelian group, Proc. **19** (1943), 366–372.
3) L. Pontrjagin, Topological Groups, Princeton, 1939.
4) J. Schreier and S. M. Ulam, Fund. Math., **24** (1935), 302–304.

if every finite factor group G/H of G by an open-and-closed subgroup H of G is a cyclic group.

Let us next consider a connected compact monothetic[1] group G which is of one dimension. Then the discrete character group G^* of G has no element of finite order, and any two elements of G^* are dependent on each other, i. e., for any two elements a^* and b^* of G^*, there exist two integers m and n such that $ma^*+nb^*=0^*$ (0^* denotes the zero element of G^*). Hence G^* is algebraically isomorphic with the additive group $RA(n_1, n_2, \dots)$ of all rational numbers r of the form $r=m/n_1 \cdots n_k$, where $\{n_k \mid k=1, 2, \dots\}$ is an arbitarary sequence of positive integers. From this follows

Theorem 11. Every connected compact monothetic group of one dimension can be obtained as a limit group of a sequence $\{K_{n_1,\dots,n_k} \mid k=1, 2, \dots\}$ of compact abelian groups $K_{n_1,\dots,n_k}$ each of which is topologically isomorphic with the additive group K of all real numbers mod 1. with the usual compact topology, where $\{n_k \mid k=1, 2, \dots\}$ is a sequence of positive integers and the homomorphism $x_{k+1} \to x_k$ of $K_{n_1,\dots,n_k,n_{k+1}}$ onto $K_{n_1,\dots,n_k}$ is given by $x_k=n_{k+1}x_{k+1}$ (mod. 1).

This limit group is denoted by $K(n_1, n_2, \dots)$. Thus every connected compact monothetic group of one dimension is separable. Further, $K(n_1, n_2, \dots)$ contains a one-parameter subgroup $L'=\{a(t) \mid -\infty < t < \infty\}$ which is dense in $K(n_1, n_2, \dots)$. Thus $K(n_1, n_2, \dots)$ is a *solenoidal group* in a sense to be defined later (§ 6), and there exists essentially only one one-parameter subgroup of $K(n_1, n_2, \dots)$ which is dense in $K(n_1, n_2, \dots)$, i. e. for any one-parameter subgroup $L''=\{b(t) \mid -\infty < t < \infty\}$ of $K(n_1, n_2, \dots)$ which is dense in $K(n_1, n_2, \dots)$, there exists a constant λ such that $a(t)=b(\lambda t)$ for all t. Finally, $K(n_1, n_2, \dots)$ is a continuous homomorphic image of $K(m_1, m_2, \dots)$ if and only if, for any k, there exists an l such that $n_1 \cdots n_k$ divides $m_1 \cdots m_l$. Thus $n_k=k$, $k=1, 2, \dots$, gives a connected compact monothetic group of one dimension $G=K(1, 2, \dots)$ of one dimension such that every connected compact monothetic group of one dimension is a continuous image of G. $K(1, 2, \dots)$ is called the *universal one-dimensional compact solenoid.*

From Theorem 4 follows

Theorem 12. There exists a compact monothetic group G such that every compact monothetic group is a continuous homomorphic image of G in the sense of Theorem 4.

This group is nothing but the compact character group $(RE1)^*=K^{(d)*}$ of the additive group $RE1=K^{(d)}$ of all real numbers mod. 1 with discrete topology, and is called the *universal compact monothetic group.*

Since $RE1=K^{(d)}$ is algebraically isomorphic with the restricted infinite direct sum of one $RA1$ (=the additive group of all rational

1) We do not need the assumption that G is monothetic. In fact, as was shown in Theorem 9, every connected compact abelian group G is monothetic if the cardinal number $\mathfrak{p}(G)$ of G does not exceed $2^{\mathfrak{c}}$, and this is really the case if G is one-dimensional (G is even separable and $\mathfrak{p}(G)=\mathfrak{c}$ in this case).

numbers with discrete topology) and a continuum number of RA (=the additive group of all rational numbers with discrete topology):

$$(13) \qquad RE1=RA1 \oplus \textstyle\sum_c RA ,$$

so we see that the universal compact monothetic group $G=(RE1)^*=K^{(d)*}$ is topologically isomorphic with the non-restricted infinite direct sum of one universal monothetic Cantor group $C(1,2,\ldots)$ and a continuum number of universal one-dimensional compact solenoids $K(1,2,\ldots)$ with the usual weak topology of Tychonoff:

$$(14) \qquad (RE1)^*=C(1,2,\ldots) \oplus \textstyle\sum_c \oplus K(1,2,\ldots) .$$

This gives the structure of the universal compact monothetic group.

§ 6. *Solenoidal groups.* A topological group G is a *solenoid*[1] or a *solenoidal group* if there exists a one-parameter subgroup $L'=\{a(t) \mid -\infty < t < \infty\}$ of G which is dense in G, or more precisely, if there exists a continuous homomorphism $t \rightarrow a(t)$ of the additive group $L=\{t \mid -\infty < t < \infty\}$ of all real numbers with the usual locally compact topology onto a topological subgroup $L'=\{a(t) \mid -\infty < t < \infty\}$ of G which is everywhere dense in G. A solenoidal group is obviously abelian and connected. Exactly as in Theorem 5, we may prove

Theorem 13. A locally compact solenoidal group is either compact or topologically isomorphic with the additive group L of all real numbers with the usual locally compact topology.

Remark 6. It is not difficult to see that there exists a complete metric solenoidal group which is not locally compact. An example of such a group may be obtained in exactly the same way as in Remark 5.

Further, since the locally compact character group L^* of L is topologically isomorphic with L itself, so we see from Theorem 2:

Theorem 14. In order that a compact abelian group G be solenoidal, it is necessary and sufficient that the discrete character group G^* of G be algebraically isomorphic with an algebraical subgroup of the additive group $RE=L^{(d)}$ of all real numbers with discrete topology.

From Theorem 3 follows:

Theorem 15. Every compact solenoidal group G can be obtained in the following way: take the additive group $L=\{t \mid -\infty < t < \infty\}$ of all real numbers with the usual locally compact topology, and introduce on it a weaker uniform structure by means of a family $F^*=\{f^*(t)\}$ of complex-valued Bohr almost periodic functions $f^*(t)$ defined on L, or by means of a family $X^*=\{a_\lambda^*\}$ of continuous characters a_λ^* on L, where $-\infty < \lambda < \infty$ and $(t, a_\lambda^*)=\lambda t$ (mod. 1). Then G is obtained by completing L with respect to this weaker uniform structure. G is separable if and only if $X^*(F^*)=\bigcup_{f^* \in F^*} X^*(f^*)$ or X^* is a countable set, where $X^*(f^*)$ is a countable set of characters for which the Fourier coefficient of $f^*(t)$ does not vanish; and so there exists a compact solenoidal group which is not separable.

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1) Cf. D. van Dantzig, Fund. Math. **15** (1930), 102–125.

We now need.

Lemma 2. Let G^* be a discrete abelian group whose cardinal number $\mathfrak{p}(G^*)$ does not exceed $\mathfrak{c}$. Then, in order that G^* be algebraically isomorphic with an algebraic subgroup of the additive group $RE=L^{(d)}$ of all real numbers with discrete topology, it is necessary and sufficient that no element of G^* be of finite order.

The condition of Lemma 2 is clearly necessary. That it is also sufficient may be proved by constructing a required isomorphism by transfinite induction.

Theorem 16. In order that a compact abelian group G be solenoidal, it is necessary and sufficient that the cardinal number $\mathfrak{p}(G)$ of G do not exceed $2^{\mathfrak{c}}$ and further that G be connected.

Theorem 16 follows from Theorem 14 and Lemma 2 if we observe the following two facts: (i) $\mathfrak{p}(G^*)\leqq\mathfrak{c}$ is equivalent with $\mathfrak{p}(G)\leqq 2^{\mathfrak{c}}$, (ii) a compact abelian group G is connected if any only if the discrete character group G^* of G has no element of finite order.

Exactly as in Theorem 12 we may prove

Theorem 17. There exists a compact solenoidal group G such that every compact solenoidal group is a continuous homomorphic image of G in the sense of Theorem 4.

This group is nothing but the compact character group of the additive group $RE=L^{(d)}$ of all real numbers with discrete topology, and is called the *universal compact solenoid.*

Since $RE=L^{(d)}$ is algebraically isomorphic with the restricted infinite direct sum of a continuum number of RA(=the additive group of all rational numbers with discrete topology):

$$RE=\sum_{\mathfrak{c}}\oplus RA\,, \tag{15}$$

so we see that the universal compact solenoid $G=(RE)^*$ is topologically isomorphic with the non-restricted infinite direct sum of a continuum number of universal one-dimensional compact solenoids $K(1, 2, \dots)$ with the usual weak topology:

$$(RE)^*=\sum_{\mathfrak{c}}\circledast K(1, 2, \dots)\,. \tag{16}$$

As we have seen in Theorems 8 and 14, every connected compact abelian group G whose cardinal number $\mathfrak{p}(G)$ does not exceed $2^{\mathfrak{c}}$ is at the same time monothetic and solenoidal. It is, however, not true that if $L'=\{a(t)\,|\,-\infty<t<\infty\}$ is a one-parameter subgroup of G which is dense in G, then every element $a(t)$ $(t\neq 0)$ of L' is a generating element of G. On the contrary, on every one-parameter subgroup L' of a compact abelian group G, there exists an element of G which is not a generating element of G. In fact, take any continuous character $\chi(a)$ on G, not identically zero, and then take a t_0 such that $\chi\big(a(t_0)\big)$ is a rational number. Then it is clear that $a(t_0)$ cannot be a generating element of G. We may, however, prove the following

Theorem 18. Let G be a compact separable solenoidal group, and let $L'=\{a(t)\,|-\infty<t<\infty\}$ be a one-parameter subgroup of G which *is dense in G. Then, except for a countable number of values of t, $a(t)$ is a generating element of G.*

Remark 7. Theorem 18 ceases to be true if a compact solenoid G is not separable. In fact, let us consider the case when G is the universal compact solenoid and let $L'=\{a(t)\,|\,-\infty<t<\infty\}$ be the dense one-parameter subgroup of G by means of which the group G was defined in the beginning. From the construction we see easily that for any real number λ there exists a uniquely determined continuous character $\chi_\lambda(a)$ defined on G such that $\chi_\lambda\big(a(t)\big)=\lambda t$ (mod. 1) for $-\infty<t<\infty$. We claim that, for any fixed t_0, $a(t_0)$ cannot be a generating element of G. In fact, let λ be a real number such that $\lambda t_0=1$; then the corresponding character $\chi_\lambda(a)$ on G satisfies $\chi_\lambda\big(a(t_0)\big)\equiv 0$ (mod. 1), and this shows that $a(t_0)$ cannot be a generating element of G.

Thus we arrived at a strange phenomenon that in the universal compact solenoid G there exists a one-parameter subgroup $L'=\{a(t)\,|\,-\infty<t<\infty\}$ which is dense in G and yet no element $a(t_0)$ of L' is a generating element of G. On the other hand, since G is connected and since $\mathfrak{p}(G)\leqq 2^{\mathfrak{c}}$, there surely exists by Theorem 9 a generating element in $G-L'$; it is even possible to find a one-parameter subgroup L'' of G on which there are infinitely many (and even continuum many) generating elements of G.

This fact shows that when two one-parameter subgroups L' and L'' of a compact solenoidal group G are given, L' and L'' are not always conjugate with each other, i. e. there does not always exist a homeomorphic automorphism of G onto itself which carries L' onto L'' over. The analogous phenomenon may also happen for a pair of generating elements of a compact monothetic group; namely, it is possible, to find a compact monothetic group G and a pair of generating elements a and a' of G such that there is no homeomorphic automosphism of G onto itself which carries a onto a' over. It is, however, to be noticed that such a phenomenon never happens for the universal monothetic Cantor group $C(1,2,\dots)$ and for the universal one-dimensional compact solenoid $K(1,2,\dots)$.

This fact suggests that, given two locally compact abelian groups H and G, it would be an interesting problem to investigate in more detail the way in which H is continuously and homomorphically mapped onto a dense subgroup H' of G. These and the related problems will be discussed on another occasion.