13. On the Ergodicity of a Certain Stationary Process*.

By Kiyosi ITÔ.

Mathematical Institute, Nagoya Imperial University.

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Let $x_t(\omega)$ be any strictly stationary process[1]. The probability law of the process $x_t(\omega)$ is a probability distribution on R^R which is invariant by the mapping T_τ that transforms $f(t)\in R^R$ into $f(t+\tau)\in R^R$ for any τ. We shall say that the process $x_t(\omega)$ is ergodic in the (strongly) mixing type if it is the case with the group of the measure-preserving mappings $\{T_\tau\}$[2]. We shall establish the

Theorem. Let $x_t(\omega)$ be any strictly[3] *stationary process of Gaussian type*[4] *with the correlation function $\rho(\tau)\equiv\int_{-\infty}^{\infty}e^{i\lambda\tau}F(d\lambda)$*[5]. *The sufficient condition that $x_t(\omega)$ should be ergodic in the (strongly) mixing type is that the spectral measure F is absolutely continuous.*

Proof. It is sufficient to show the identity:

$$(1)\qquad \lim_{\tau\to\infty}P\{(x_{s_1},x_{s_2},\dots,x_{s_m})\in E_m,\ (x_{t_1+\tau},x_{t_2+\tau},\dots,x_{t_n+\tau})\in E_n\}$$

or

$$\lim_{\tau\to\infty}P\{(x_{s_1},x_{s_2},\dots,x_{s_m},x_{t_1+\tau},x_{t_2+\tau},\dots,x_{t_n+\tau})\in E_m\otimes E_n\}$$
$$=P\{(x_{s_1},x_{s_2},\dots,x_{s_m})\in E_m\}P\{(x_{t_1},x_{t_2},\dots,x_{t_n})\in E_n\}$$

where E_m and E_n are any bounded Borel sets respectively in R^m and in R^n and $s_1<s_2<\cdots<s_m$, $t_1<t_2<\cdots<t_n$. We may assume $\mathscr{E}(x_t)=0$ and $\mathscr{E}(x_t^2)=1$ with no loss of generality.

If u_i, $i=1,2,\dots,p$, are all different, the matrix $\{\rho(u_i-u_j)\,;\ i,j=1,2,\dots,p\}$ is strictly positive definite, that is $\sum_{i,j}\rho(u_i-u_j)\xi_i\bar{\xi}_j>0$ for any system ξ_i, $i=1,2,\dots,p$, such that $\sum_i|\xi_i|^2\neq 0$. In fact we have

$$(2)\qquad \sum_{i,j}\rho(u_i-u_j)\xi_i\bar{\xi}_j=\int_{-\infty}^{\infty}\Big|\sum_k e^{i\lambda u_k}\xi_k\Big|^2F(d\lambda)\geqq 0\,.$$

If the last equality holds, we shall have $\sum_k e^{i\lambda u_k}\xi_k=0$ for any spectrum of F. Since F is absolutely continuous, the set of all the spectra of F has accumulation points $\neq\infty$. Therefore $\sum_k e^{i\lambda u_k}\xi_k$, as an integral

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1) Cf. A. Khintchine: Korrelationstheorie der stationären stochastischen Prozesse (Math. Ann. 109).

2) Cf. E. Hopf: Ergodentheorie (Erg. d. Math.) 1937, p. 36, Def. 11.1.

3) The condition "strictly" can be omitted since any weakly stationary process of Gaussian type is strictly stationary.

4) Cf. A. Khintchine, loc. cit. 1), the remark at the end of § 2.

5) The correlation function of any stationary process can be always expressible in this form. Cf. A. Khintchine, loc. cit. 1).

function of λ, is identically equal to 0. Thus we should have $\xi_k=0$, $k=1,2,\ldots,p$, contrary to the assumption. Therefore we have $\sum_{i,j}\rho(u_i-u_j)\xi_i\bar{\xi}_j>0$.

The probability law of $(x_{s_1}, x_{s_2}, \ldots, x_{s_m}, x_{t_1+\tau}, x_{t_2+\tau}, \ldots, x_{t_n+\tau})$ is a normalized $(m+n)$-dimensional Gaussian distribution with the correlation matrix:

$$M(\tau)\equiv\begin{pmatrix} S & R^*(\tau) \\ R(\tau) & T \end{pmatrix}, \tag{3}$$

where S and T are respectively the correlation matrices of $(x_{s_1}, x_{s_2}, \ldots, x_{s_m})$ and of $(x_{t_1}, x_{t_2}, \ldots, x_{t_n})$ and the elements $r_{ij}(\tau)$, $i=1,2,\ldots,n$, $j=1,2,\ldots,m$ of $R(\tau)$ are equal to $\rho(t_i-s_j+\tau)$, and $R^*(\tau)$ is the transposed matrix of $R(\tau)$.

We may suppose that $s_1, s_2, \ldots, s_m$, $t_1+\tau, t_2+\tau, \ldots, t_n+\tau$ are all different for a sufficiently large τ. So $M(\tau)$ is a strictly positive definite matrix. The probability law of $(x_{s_1}, x_{s_2}, \ldots, x_{s_m}, x_{t_1+\tau}, x_{t_2+\tau}, \ldots, x_{t_n+\tau})$ is

$$\frac{1}{(2\pi)^{\frac{m+n}{2}}\sqrt{\mathrm{Det.}\, M(\tau)}}\, e^{-\frac{1}{2}(M(\tau)^{-1}\xi,\, \xi)} \qquad (\xi\in R^{m+n}). \tag{4}$$

As F is absolutely continuous, $r_{ij}(\tau)\equiv\int_{-\infty}^{\infty}e^{i\lambda(t_i-s_j+\tau)}F(d\lambda)$ tends to 0 on account of the Riemann-Lebesgue theorem. Therefore we have

$$M(\tau)\rightarrow\begin{pmatrix} S & 0 \\ 0 & T \end{pmatrix}, \qquad M(\tau)^{-1}\rightarrow\begin{pmatrix} S^{-1} & 0 \\ 0 & T^{-1} \end{pmatrix}. \tag{5}$$

since Det. T and Det. S do not vanish. Therefore the expression (4) converges to

$$\frac{1}{(2\pi)^{\frac{m}{2}}\sqrt{\mathrm{Det.}\, S}}\, e^{-\frac{1}{2}(S^{-1}\eta,\, \eta)}\, \frac{1}{(2\pi)^{\frac{n}{2}}\sqrt{\mathrm{Det.}\, T}}\, e^{-\frac{1}{2}(T^{-1}\zeta,\, \zeta)} \tag{6}$$

uniformly as far as (η, ζ) runs over a bounded region in R^{m+n}. The factors in (6) are clearly the probability laws of $(x_{s_1}, x_{s_2}, \ldots, x_{s_m})$ and of $(x_{t_1}, x_{t_2}, \ldots, x_{t_n})$ respectively. Thus the identity (1) can be deduced at once.