PAPERS COMMUNICATED

12. Projective Parameters in Projective and Conformal Geometries.

By Kentaro YANO.

Mathematical Institute, Tokyo Imperial University.

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§ 1. *Projective parameters in projective geometry.*

In an n-dimensional space A_n with the affine connection Γ^i_{jk}, a system of curves called paths is defined by the differential equations of the form

$$(1.1) \qquad \frac{d^2x^i}{ds^2}+\Gamma^i_{jk}\frac{dx^j}{ds}\frac{dx^k}{ds}=0 \qquad (i, j, k, \dots=1, 2, \dots, n)$$

as autoparallel curves, where s is called affine parameter on each path. Conversely, if we are given the differential equations of the form (1.1) in an n-dimensional space X_n, we can define a symmetric affine connection in this space taking Γ^i_{jk} as the components of the connection. The study of the properties of these differential equations constitutes the affine geometry of paths[1]. But, an affine connection is not defined uniquely by the system of paths (1.1). H. Weyl[2] and L. P. Eisenhart[3] have independently shown that any two affine connections whose components $\bar{\Gamma}^i_{jk}$ and Γ^i_{jk} are related by the equations of the form

$$(1.2) \qquad \bar{\Gamma}^i_{jk}=\Gamma^i_{jk}+\delta^i_j\psi_k+\delta^i_k\psi_j\,,$$

where ψ_j are components of an arbitrary covariant vector not necessarily gradient, give the same paths. In this sense, the change over from $\bar{\Gamma}^i_{jk}$ to Γ^i_{jk} is called the projective change of affine connections, and the study of those properties which are invariant under such changes of affine connections is called the projective geometry of paths[4].

To study the projective geometry of paths, T. Y. Thomas[5] has introduced the functions

$$(1.3) \qquad \Pi^i_{jk}=\Gamma^i_{jk}-\frac{1}{n+1}(\delta^i_j\Gamma^a_{ak}+\delta^i_k\Gamma^a_{aj})\,,$$

which are invariant under projective change of affine connections (1.2).

1) L. P. Eisenhart and O. Veblen: The Riemann geometry and its generalisation. Proc. Nat. Acad. Sci. **8** (1922), pp. 19–23.

2) H. Weyl: Zur Infinitesimalgeometrie: Einordnung der projektiven und der konformen Auffassung. Göttinger Nachrichten (1921), pp. 99–112.

3) L. P. Eisenhart: Spaces with corresponding paths. Proc. Nat. Acad. Sci. **8** (1922), pp. 233–238.

4) O. Veblen: Projective and affine geometry of paths. ibidem, pp. 347–350.

5) T. Y. Thomas: On the projective and equi-projective geometries of paths. ibidem, **11** (1925), pp. 199–203.

Although Π^i_{jk} are invariant under the projective change of affine connections, their law of transformation under the change of coordinates is not identical with that of the components of the affine connection, that is the case when and only when the jacobian of the transformation is constant. The study of the invariant properties of Π^i_{jk} under such restricted transformations constitutes the equi-projective geometry of paths[1].

To avoid this inconvenience, T. Y. Thomas[2] has introduced an extra dimension x^0 and defined an affine connection ${}^*\Pi^\lambda_{\mu\nu}$ $(\lambda, \mu, \nu, \dots = 0, 1, 2, \dots, n)$ in an associated space of $(n+1)$ dimensions by means of the relations

$$(1.4)\qquad {}^*\Pi^\lambda_{0\nu} = {}^*\Pi^\lambda_{\nu 0} = -\frac{1}{n+1}\delta^\lambda_\nu\,, \quad {}^*\Pi^i_{jk} = \Pi^i_{jk}\,, \quad {}^*\Pi^0_{jk} = \frac{n+1}{n-1}\Pi_{jk}\,,$$

where

$$\Pi_{jk} = \Pi^i_{\cdot jki} \quad \text{and} \quad \Pi^i_{\cdot jkh} = \frac{\partial \Pi^i_{jk}}{\partial x^h} - \frac{\partial \Pi^i_{jh}}{\partial x^k} + \Pi^a_{jk}\Pi^i_{ah} - \Pi^a_{jh}\Pi^i_{ak}\,,$$

and formulated the projective geometry of paths as the invariant theory of the affine connection of this $(n+1)$-dimensional associated space under the special change of coordinates

$$(1.5)\qquad x^{0'} = x^0 + \log\left|\frac{\partial x'}{\partial x}\right|, \qquad x^{i'} = x^{i'}(x^1, x^2, \dots, x^n)\,.$$

This idea of introducing an extra coordinate x^0 is adopted later by O. Veblen[3]. O. Veblen has defined the projective geometry as the invariant theory of $\Pi^\lambda_{\mu\nu}$, symmetric and satisfying the following conditions

$$(1.6)\qquad \Pi^\lambda_{0\nu} = \Pi^\lambda_{\nu 0} = \delta^\lambda_\nu\,, \qquad \frac{\partial}{\partial x^0}\Pi^\lambda_{\mu\nu} = 0\,,$$

under special transformations of coordinates

$$(1.7)\qquad x^{0'} = x^0 + \log\rho\ (x^1, x^2, \dots, x^n)\,, \qquad x^{i'} = x^{i'}(x^1, x^2, \dots, x^n)\,.$$

Let $\Pi^\lambda_{\mu\nu}$ be the components of an affine connection in an $(n+1)$-dimensional space A_{n+1}. If there exists a coordinate system in which the components of connection $\Pi^\lambda_{\mu\nu}$ satisfy the conditions (1.6), then the A_{n+1} referred to this coordinate system may be taken to represent a projective space P_n. From this point of view, J. H. C. Whitehead[4] has studied the representation of projective spaces, and derived many interesting results on generalized projective geometry.

Let Π^i_{jk} be the components of an symmetric affine connection of an n-dimensional space A_n, then introducing a symmetric tensor Π^0_{jk},

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1) T. Y. Thomas: On the equi-projective geometry of paths. ibidem, pp. 592–594.

2) T. Y. Thomas: A projective theory of affinely connected manifolds. Math. Zeitschr. **25** (1926), pp. 723–733.

3) O. Veblen: Generalized projective geometry. Journal of the London. Math. Soc. **4** (1929), pp. 140–160.

4) J. H. C. Whitehead: The representation of projective spaces, Annals of Math, **32** (1931), pp. 327–360.

we can construct a symmetric affine connection $\Pi^{\lambda}_{\mu\nu}$ of an $(n+1)$-dimensional space A_{n+1} by means of Π^{i}_{jk}, Π^{0}_{jk} and $\Pi^{\lambda}_{0\nu}=\Pi^{\lambda}_{\nu 0}=\delta^{\lambda}_{\nu}$.

The equations of paths in A_{n+1} are given by

$$\frac{d^2x^\lambda}{dt^2}+\Pi^{\lambda}_{\mu\nu}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt}=0, \tag{1.8}$$

where t is an affine parameter for the paths of A_{n+1}. Putting $\lambda=0$ and $\lambda=i$ in (1.8), we obtain

$$\begin{cases} \dfrac{d^2x^0}{dt^2}+\Pi^{0}_{jk}\dfrac{dx^j}{dt}\dfrac{dx^k}{dt}+\left(\dfrac{dx^0}{dt}\right)^2=0 \quad \text{and} \\ \dfrac{d^2x^i}{dt^2}+\Pi^{i}_{jk}\dfrac{dx^j}{dt}\dfrac{dx^k}{dt}+2\left(\dfrac{dx^0}{dt}\right)\dfrac{dx^i}{dt}=0 \end{cases} \tag{1.9}$$

respectively. If we introduce a new parameter s by means of the relation $2\left(\frac{dx^0}{dt}\right)=\frac{d^2t}{ds^2}\Big/\left(\frac{dt}{ds}\right)^2$ or $x^0=\frac{1}{2}\log\frac{dt}{ds}$, the equations (1.9) take respectively the form

$$\{t, s\}=-2\Pi^{0}_{jk}\frac{dx^j}{ds}\frac{dx^k}{ds} \quad \text{and} \quad \frac{d^2x^i}{ds^2}+\Pi^{i}_{jk}\frac{dx^j}{ds}\frac{dx^k}{ds}=0, \tag{1.10}$$

where $\{t, s\}$ denotes the Schwarzian derivative of t with respect to s.

Thus, the projective connection of O. Veblen and J. H. C. Whitehead defines a system of paths in A_n and a projective parameter t on each path. The projective parameter t introduced first by J. H. C. Whitehead plays an important part in the study of projective geometry of paths.

L. Berwald[1] developed, on the other hand, the projective geometry of paths resting on two notions: the notion of the class of affine connections belonging to a system of paths and the notion of a projective parameter of a system of paths. He defines the paths by (1.1) and projective parameter by

$$\{t, s\}=-2\Gamma^{0}_{jk}\frac{dx^j}{ds}\frac{dx^k}{ds}. \tag{1.11}$$

The projective parameter t being defined by a Schwarzian derivative,

(a) it is determined, up to an arbitrary linear fractional transformation, on each path of the system at the same time,

he requires moreover that

(b) it is not altered by transformations of coordinates,

(c) it remains the same for all affine connections of the class belonging to the system of paths.

From the condition (b) and (1.11), we know that Γ^{0}_{jk} are the components of a tensor, and from the condition (c), we conclude that, the

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1) L. Berwald: On the projective geometry of paths. Annals of Math. **37** (1936), pp. 879–898.

law of transformation of Γ^0_{jk} under the projective change of affine connections (1.2) is

$$\bar{\Gamma}^0_{jk}=\Gamma^0_{jk}+\frac{1}{2}\left(\frac{\partial\psi_j}{\partial x^k}+\frac{\partial\psi_k}{\partial x^j}\right)-\psi_i\Gamma^i_{jk}-\psi_j\psi_k\,. \tag{1.12}$$

The term $\frac{1}{2}\left(\frac{\partial\psi_j}{\partial x^k}+\frac{\partial\psi_k}{\partial x^j}\right)$ appears owing to the fact that Γ^0_{jk} in (1.11) are the coefficients of a quadratic form and consequently only the symmetric part of Γ^0_{jk} is in question.

The present author[1] has shown that all these projective geometries of paths may be naturally interpretated from the standpoint of E. Cartan[2]. If we define Cartan's projective connection by the formulae

$$dA_0=dx^iA_i\,,\qquad dA_j=\Gamma^0_{jk}dx^kA_0+\Gamma^i_{jk}dx^kA_i \tag{1.13}$$

and paths as the curves whose developments in tangent projective space are straight lines, then the equation of paths may be written as

$$\frac{d^2}{dt^2}\rho A_0=0\,, \tag{1.14}$$

and the differential equations of the paths coincide with (1.1) and t is precisely the projective parameter defined by (1.11). The theory of T. Y. Thomas is obtained if we adopt the so-called repère naturel in the space with normal projective connection.

J. Haantjes[3] studied the projective geometry of paths using the homogeneous coordinates of D. van Dantzig[4]. The differential equations of the paths are

$$\frac{d^2x^\lambda}{dr^2}+\Pi^\lambda_{\mu\nu}\frac{dx^\mu}{dr}\frac{dx^\nu}{dr}=\alpha x^\lambda+\beta\frac{dx^\lambda}{dr}\,, \tag{1.15}$$

where x^λ are homogeneous coordinates, $\Pi^\lambda_{\mu\nu}$ components of the symmetric projective connection which are homogeneous functions of degree -1 and satisfy $\Pi^\lambda_{\mu\nu}x^\mu=0$, and r an arbitrary parameter on the paths. J. Haantjes also introduced special homogeneous coordinates u^0 and u^1 on each paths and showed that their ratio coincides with the projective parameter t appeared in old theories.

It is shown in the author's These that if we choose a suitable factor ρ and parameter t on each path, the equations of paths (1.15) may be written as

$$\frac{d^2\rho x^\lambda}{dt^2}+\Pi^\lambda_{\mu\nu}(\rho x)\frac{d\rho x^\mu}{dt}\frac{d\rho x^\nu}{dt}=0\,. \tag{1.16}$$

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1) K. Yano: Les espaces à connexion projective et la géométrie projective des paths. Thèse, Paris, (1938).

2) É. Cartan: Leçons sur la théorie des espaces à connexion projective. Paris, Gauthier-Villars, (1937).

3) J. Haantjes: On the projective geometry of paths. Proc. Edinburgh Math. Soc. **5** (1937), pp. 103–115.

4) D. van Dantzig: Theorie des projektiven Zusammenhangs n-dimensionaler Räume. Math. Ann. **106** (1932), pp. 400–454.

It is interesting to remark that, in all these theories, the equations of paths in their most simple forms (1.8), (1.14) and (1.16) have the same form that the second derivatives of the coordinates vanish, and that the parameter t which admits this simplification is the projective parameter.

§ 2. *Projective parameters in conformal geometry.*

In this section, we shall show that the method of L. Berwald explained in § 1 may be applied also to the conformal geometry.

(i) *Conformal geometry.*

Let us consider an n-dimensional Riemann space V_n with the fundamental quadratic form $ds^2 = g_{jk}dx^j dx^k$. By a conformal transformation $\bar{g}_{jk} = \rho^2 g_{jk}$ of the fundamental tensor, the line-element ds of each curve and the Christoffel symbols $\{{}^{\,i}_{jk}\}$ are transformed into $d\bar{s}$ and $\overline{\{{}^{\,i}_{jk}\}}$ respectively by the formulae of the form

$$d\bar{s} = \rho ds, \tag{2.1}$$

$$\overline{\{{}^{\,i}_{jk}\}} = \{{}^{\,i}_{jk}\} + \delta^i_j \rho_k + \delta^i_k \rho_j - g^{ia}\rho_a g_{jk}, \qquad \left(\rho_j = \frac{\partial}{\partial x^j} \log \rho\right) \tag{2.2}$$

consequently the tangent vector $\dfrac{dx^i}{ds}$ and the curvature vector $\dfrac{\delta}{\delta s}\dfrac{dx^i}{ds} = \dfrac{\delta^2 x^i}{\delta s^2}$ of any curve are transformed into $\dfrac{dx^i}{d\bar{s}}$ and $\dfrac{\delta^2 x^i}{\delta \bar{s}^2}$ respectively by

$$\frac{dx^i}{d\bar{s}} = \frac{dx^i}{ds} \Big/ \frac{d\bar{s}}{ds} \tag{2.3}$$

$$\frac{\delta^2 x^i}{\delta \bar{s}^2} = \frac{\delta^2 x^i}{\delta s^2} \Big/ \left(\frac{d\bar{s}}{ds}\right)^2 + \frac{dx^i}{ds}\frac{d^2\bar{s}}{ds^2} \Big/ \left(\frac{d\bar{s}}{ds}\right)^3 - g^{ia}\rho_a \Big/ \left(\frac{d\bar{s}}{ds}\right)^2, \tag{2.4}$$

where $\delta/\delta s$ denotes the covariant differentiation along the curve with respect to the Christoffel symbols $\{{}^{\,i}_{jk}\}$ and $\delta/\delta\bar{s}$ with respect to $\overline{\{{}^{\,i}_{jk}\}}$.

Now we define a projective parameter t on each curve $x^i(s)$ by means of the Schwarzian differential equation of the form

$$\{t, s\} = \frac{1}{2} g_{jk} \frac{\delta^2 x^j}{\delta s^2} \frac{\delta^2 x^k}{\delta s^2} - \Pi^0_{jk} \frac{dx^j}{ds} \frac{dx^k}{ds}. \tag{2.5}$$

t being defined on each curve up to an arbitrary linear fractional transformation, we assume moreover that it is not altered by transformations of coordinates and that it remains same for all metrics related by a conformal transformation $\bar{g}_{jk} = \rho^2 g_{jk}$. From the first assumption, we can conclude that the functions Π^0_{jk} are components of a symmetric tensor. From the second assumption, we find, by a straightforward calculation with the aid of (2.1), (2.3), (2.4) and

$$\{t, \bar{s}\} = (\{t, s\} - \{\bar{s}, s\}) \Big/ \left(\frac{d\bar{s}}{ds}\right)^2,$$

that the law of transformation of Π^0_{jk} under a conformal transformation $\bar{g}_{jk} = \rho^2 g_{jk}$ is

$$(2.6)\qquad \bar{\Pi}^0_{jk}=\Pi^0_{jk}+\frac{\partial\rho_j}{\partial x^k}-\rho_i\left\{{i \atop jk}\right\}-\rho_j\rho_k+\frac{1}{2}g^{ab}\rho_a\rho_b g_{jk}.$$

Thus the conformal geometry is fixed by giving g_{jk} and consequently $\left\{{i \atop jk}\right\}$ and Π^0_{jk} whose laws of transformations under the conformal transformation $\bar{g}_{jk}=\rho^2 g_{jk}$ are (2.2) and (2.6) respectively.

(ii) *Conformal circles.*

Geodesic circles[1] in Riemann geometry are defined by the equations

$$(2.7)\qquad \frac{\delta^3x^i}{\delta s^3}+g_{jk}\frac{\delta^2x^j}{\delta s^2}\frac{\delta^2x^k}{\delta s^2}\frac{dx^i}{ds}=0.$$

The left-hand side of this equation is not invariant under a conformal transformation $\bar{g}_{jk}=\rho^2 g_{jk}$. Under this transformation it is transformed by

$$(2.8)\qquad \frac{\delta^3x^i}{\delta \bar{s}^3}+\bar{g}_{jk}\frac{\delta^2x^j}{\delta \bar{s}^2}\frac{\delta^2x^k}{\delta \bar{s}^2}\frac{dx^i}{d\bar{s}}$$
$$=\frac{1}{\rho^3}\left(\frac{\delta^3x^i}{\delta s^3}+g_{jk}\frac{\delta^2x^j}{\delta s^2}\frac{\delta^2x^k}{\delta s^2}\frac{dx^i}{ds}+\rho_{jk}\frac{dx^j}{ds}\frac{dx^k}{ds}\frac{dx^i}{ds}-\rho^i_{\cdot k}\frac{dx^k}{ds}\right),$$

where

$$(2.9)\qquad \rho_{jk}=\frac{\partial\rho_j}{\partial x^k}-\rho_i\left\{{i \atop jk}\right\}-\rho_j\rho_k+\frac{1}{2}g^{ab}\rho_a\rho_b g_{jk},\quad\text{and}\quad \rho^i_{\cdot k}=g^{ia}\rho_{ak},$$

hence, from (2.6) and (2.8) we have

$$\frac{\delta^3x^i}{\delta \bar{s}^3}+\bar{g}_{jk}\frac{\delta^2x^j}{\delta \bar{s}^2}\frac{\delta^2x^k}{\delta \bar{s}^2}\frac{dx^i}{d\bar{s}}-\bar{\Pi}^0_{jk}\frac{dx^j}{d\bar{s}}\frac{dx^k}{d\bar{s}}\frac{dx^i}{d\bar{s}}+\bar{\Pi}^i_{\infty k}\frac{dx^k}{d\bar{s}}$$
$$=\frac{1}{\rho^3}\left[\frac{\delta^3x^i}{\delta s^3}+g_{jk}\frac{\delta^2x^j}{\delta s^2}\frac{\delta^2x^k}{\delta s^2}\frac{dx^i}{ds}-\Pi^0_{jk}\frac{dx^j}{ds}\frac{dx^k}{ds}\frac{dx^i}{ds}+\Pi^i_{\infty k}\frac{dx^k}{ds}\right],$$

where $\bar{\Pi}^i_{\infty k}=\bar{g}^{ij}\bar{\Pi}^0_{jk}$ and $\Pi^i_{\infty k}=g^{ij}\Pi^0_{jk}$,

which shows that the curve defined by

$$(2.10)\qquad \frac{\delta^3x^i}{\delta s^3}+g_{jk}\frac{\delta^2x^j}{\delta s^2}\frac{\delta^2x^k}{\delta s^2}\frac{dx^i}{ds}-\Pi^0_{jk}\frac{dx^j}{ds}\frac{dx^k}{ds}\frac{dx^i}{ds}+\Pi^i_{\infty k}\frac{dx^k}{ds}=0$$

has conformal property. This is the conformal circle found by the author in the conformally connected space[2].

(iii) *Conformal curvature tensor.*

It is well known that the Riemann-Christoffel curvature tensor

$$R^i_{\cdot jkh}=\frac{\partial\left\{{i \atop jk}\right\}}{\partial x^h}-\frac{\partial\left\{{i \atop jh}\right\}}{\partial x^k}+\left\{{a \atop jk}\right\}\left\{{i \atop ah}\right\}-\left\{{a \atop jh}\right\}\left\{{i \atop ak}\right\}$$

is transformed into $\bar{R}^i_{\cdot jkh}$ under a conformal transformation $\bar{g}_{jk}=\rho^2 g_{jk}$ of the fundamental metric tensor g_{jk} by the formulae

1) K. Yano: Concircular geometry I. Proc. **16** (1940), pp. 195–200.

2) K. Yano: Sur les circonférences généralisées dans les espaces à connexion conforme, Proc. **14** (1938), pp. 329–332.

(2.11) $$\bar{R}^{i}{}_{jkh}=R^{i}_{\cdot jkh}-\rho_{jk}\delta^{i}_{h}+\rho_{jh}\delta^{i}_{k}-g_{jk}\rho^{i}_{\cdot h}+g_{jh}\rho^{i}_{\cdot k}\,.$$

Substituting the relation

$$\rho_{jk}=\bar{\Pi}^{0}_{jk}-\Pi^{0}_{jk}$$

obtained from (2.6) in the equation (2.11), we have

$$\begin{aligned}\bar{R}^{i}_{\cdot jkh}+\bar{\Pi}^{0}_{jk}\delta^{i}_{h}-\bar{\Pi}^{0}_{jh}\delta^{i}_{k}+\bar{g}_{jk}\bar{\Pi}^{i}_{\infty h}-\bar{g}_{jh}\bar{\Pi}^{i}_{\infty k}\\ =R^{i}_{\cdot jkh}+\Pi^{0}_{jk}\delta^{i}_{h}-\Pi^{0}_{jh}\delta^{i}_{k}+g_{jk}\Pi^{i}_{\infty h}-g_{jh}\Pi^{i}_{\infty k}\,,\end{aligned}$$

which shows that the curvature tensor defined by

(2.12) $$C^{i}_{\cdot jkh}=R^{i}_{\cdot jkh}+\Pi^{0}_{jk}\delta^{i}_{h}-\Pi^{0}_{jh}\delta^{i}_{k}+g_{jk}\Pi^{i}_{\infty h}-g_{jh}\Pi^{i}_{\infty k}$$

is a conformal invariant.

(iv) *Determination of* Π^{0}_{jk} *in terms of* g_{jk}.

Since the curvature tensor $C^{i}_{\cdot jkh}$ is a conformal one, the condition

(2.13) $$C^{i}_{\cdot jki}=0$$

is also a conformal one. From (2.12) and (2.13), we have

(2.14) $$0=R_{jk}+(n-2)\Pi^{0}_{jk}+g_{jk}g^{ab}\Pi^{0}_{ab}\,,$$

from which

(2.15) $$g^{ab}\Pi^{0}_{ab}=-\frac{R}{2(n-1)}\,,$$

where $R_{jk}=R^{i}_{\cdot jki}$ and $R=g^{ab}R_{ab}$.

Substituting (2.15) in (2.14) we find

(2.16) $$\Pi^{0}_{jk}=-\frac{R_{jk}}{n-2}+\frac{Rg_{jk}}{2(n-1)(n-2)}\,.$$

Thus, the functions Π^{0}_{jk} are completely determined in terms of g_{jk} by imposing purely conformal condition (2.13). If we substitute (2.16) in (2.5), we have

(2.17) $$\{t,s\}=\frac{1}{2}g_{jk}\frac{\delta^{2}x^{j}}{\delta s^{2}}\frac{\delta^{2}x^{k}}{\delta s^{2}}+\frac{1}{n-2}R_{jk}\frac{dx^{j}}{ds}\frac{dx^{k}}{ds}-\frac{R}{2(n-1)(n-2)}\,.$$

The projective parameter defined by this equation may be called the preferred projective parameter on each curve in Riemann space.

Substituting (2.16) in (2.12), we have

(2.18) $$\begin{aligned}C^{i}_{\cdot jkh}=R^{i}_{\cdot jkh}-\frac{1}{n-2}(R_{jk}\delta^{i}_{h}-R_{jh}\delta^{i}_{k}+g_{jk}R^{i}_{\cdot h}-g_{jh}R^{i}_{\cdot k})\\ +\frac{R}{(n-1)(n-2)}(g_{jk}\delta^{i}_{h}-g_{jh}\delta^{i}_{k})\end{aligned}$$

which is the Weyl conformal curvature tensor.

(v) *The theory of T. Y. Thomas*[1].

1) T. Y. Thomas: The differential invariants of generalized spaces. Cambridge University Press, (1935).

Under a conformal transformation $\bar{g}_{jk}=\rho^2 g_{jk}$ of the fundamental tensor, the tensor density of weight $-\frac{2}{n}$ defined by

$$G_{jk}=g_{jk}/g^{\frac{1}{n}} \tag{2.19}$$

is invariant where g denotes the determinant formed with g_{jk}. Then, on each curve in V_n, a parameter σ is defined by

$$d\sigma^2=G_{jk}dx^j dx^k\,, \tag{2.20}$$

this parameter σ is a conformal one but is not a scalar and its law of transformation under the transformations of coordinates $x^{i'}=x^{i'}(x^1, x^2, \dots, x^n)$ is

$$d\sigma'=\Delta^{-\frac{1}{n}}d\sigma\,, \tag{2.21}$$

where Δ is the jacobian of the transformation: $\Delta=\left|\frac{\partial x^i}{\partial x^{i'}}\right|$.

Denoting by $\delta/\delta\sigma$ the formal covariant differentiation along the curve with the use of Christoffel symbols formed with G_{jk}:

$$^*\Pi^i_{jk}=\frac{1}{2}G^{ia}\left(\frac{\partial G_{aj}}{\partial x^k}+\frac{\partial G_{ak}}{\partial x^j}-\frac{\partial G_{jk}}{\partial x^a}\right),$$

we define the projective parameter t by

$$\{t, \sigma\}=\frac{1}{2}G_{jk}\frac{\delta^2 x^j}{\delta\sigma^2}\frac{\delta^2 x^k}{\delta\sigma^2}-{}^*\Pi^0_{jk}\frac{dx^j}{d\sigma}\frac{dx^k}{d\sigma}\,, \tag{2.22}$$

and require that t is a conformally invariant scalar under the transformation of coordinates. From this assumption we obtain the law of transformation of the functions $^*\Pi^0_{jk}$:

$$^*\Pi^0_{j'k'}=\frac{\partial x^j}{\partial x^{j'}}\frac{\partial x^k}{\partial x^{k'}}{}^*\Pi^0_{jk}+\left(\frac{\partial\psi_{j'}}{\partial x^{k'}}-\psi_{i'}{}^*\Pi^{i'}_{j'k'}\right)+\psi_{j'}\psi_{k'} -\frac{1}{2}G^{a'b'}\psi_{a'}\psi_{b'}G_{j'k'} \tag{2.23}$$

where

$$\psi_{j'}=\frac{\partial}{\partial x^{j'}}\log\Delta^{-\frac{1}{n}}.$$

$^*\Pi^0_{jk}$ being invariant under a conformal transformation $\bar{g}_{jk}=\rho^2 g_{jk}$.

The transformation law of the functions $^*\Pi^0_{jk}$ being thus obtained, we can show by a straightforward calculation that the curves defined by the differential equations

$$\frac{\delta^3 x^i}{\delta\sigma^3}+G_{jk}\frac{\delta^2 x^j}{\delta\sigma^2}\frac{\delta^2 x^k}{\delta\sigma^2}\frac{dx^i}{d\sigma}-{}^*\Pi^0_{jk}\frac{dx^j}{d\sigma}\frac{dx^k}{d\sigma}\frac{dx^i}{d\sigma}+{}^*\Pi^i_{\infty k}\frac{dx^k}{d\sigma}=0 \tag{2.24}$$

is a conformal one, where

$$^*\Pi^i_{\infty k}=G^{ij}\,{}^*\Pi^0_{jk}\,,$$

and the left-hand side of these equations are the components of a vector density and the quantities defined by

(2.25) $$^*C^{\,i}_{\cdot jkh} = {}^*\Pi^{\,i}_{\cdot jkh} + {}^*\Pi^0_{jk}\delta^i_h - {}^*\Pi^0_{jh}\delta^i_k + G_{jk}{}^*\Pi^i_{\infty h} - G_{jh}{}^*\Pi^i_{\infty k}$$

are components of a tensor, where

$$^*\Pi^{\,i}_{\cdot jkh} = \frac{\partial^*\Pi^i_{jk}}{\partial x^h} - \frac{\partial^*\Pi^i_{jh}}{\partial x^k} + {}^*\Pi^a_{jk}{}^*\Pi^i_{ah} - {}^*\Pi^a_{jh}{}^*\Pi^i_{ak}.$$

The conformal invariance of (2.24) and (2.25) being evident, we have thus defined a conformal curve and a conformal tensor. If we impose the conformal condition

(2.26) $$^*C^{\,i}_{\cdot jki} = 0$$

on $^*C^{\,i}_{\cdot jkh}$, the functions $^*\Pi^0_{jk}$ may be determined completely in terms of G_{jk}, thus

(2.27) $$^*\Pi^0_{jk} = -\frac{^*\Pi_{jk}}{n-2} + \frac{^*\Pi G_{jk}}{2(n-1)(n-2)},$$

where $$^*\Pi_{jk} = {}^*\Pi^{\,i}_{\cdot jki} \quad \text{and} \quad {}^*\Pi = G^{ab*}\Pi_{ab}.$$

It is shown by a straightforward calculation that, if we substitute (2.27) in (2.24) and (2.25), we obtain precisely (2.10) and (2.18) and we know that (2.24) defines the conformal circle and (2.25) defines the Weyl's conformal curvature tensor.