29. *Parallel Tangent Deformation, Concircular Transformation and Concurrent Vector Field.*

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1. If there is a one-to-one correspondence between the points of two curves in a three dimensional Euclidean space, and if the tangents at corresponding points are parallel, then the principal normals and consequently the binormals are parallel also. In this case, one of the two curves is said to be deducible from the other by a Combescure transformation. H. A. Hayden[1] has tried to extend this result to the case of an n-dimensional Riemannian space using the word "parallel" in Levi-Civita's sense. The parallel tangent deformation is by definition an infinitesimal transformation which displaces the tangent parallelly at each point, and the generalized Combescure transformation (briefly G. C. transformation) a parallel tangent deformation which displaces first normal, second normal, ..., and $(n-1)$-st normal parallelly also at each point.

If a curve is such that every parallel tangent deformation is a G. C. transformation, he says that the curve possesses the G. C. property, and if a space is such that every curve in it possesses the G. C. property, he says that the space itself possesses the G. C. property. Then he showed that the only Riemannian space which possesses the G. C. property is flat space.

In the Paragraph 2 of this Note, we shall consider the space in which exists a vector field that defines a parallel tangent deformation for every curve in the space, and show that such a space is the one which admits a concircular transformation[2].

In the Paragraph 3, we shall consider the relations between such space and the space which contains a concurrent vector field.

2. Let $x^\lambda(s)$ $(\lambda, \mu, \nu, \dots = 1, 2, \dots, n)$ be a curve in an n-dimensional Riemannian space V_n, s being the arc length, and let every point x^λ on the curve be displaced to $\bar{x}^\lambda = x^\lambda + \varepsilon\xi^\lambda$ by an infinitesimal deformation, ε being an infinitesimal constant. Then the curve $x^\lambda(s)$ is deformed infinitesimally and the equations of the deformed curve are

$$\bar{x}^\lambda(\bar{s}) = x^\lambda(s) + \varepsilon\xi^\lambda(s)\,. \tag{2.1}$$

Differentiation of these equations with respect to $\bar{s}$ gives

$$\frac{d\bar{x}^\lambda}{d\bar{s}} = \left(\frac{dx^\lambda}{ds} + \varepsilon\frac{d\xi^\lambda}{ds}\right)\frac{ds}{d\bar{s}}\,. \tag{2.2}$$

1) H. A. Hayden: Deformations of a curve, in a Riemannian n-space, which displaces certain vectors parallelly at each point. Proc. London Math. Soc. **32** (1931), 321–336.

2) K. Yano: Concircular geometry I, II, III, IV, Proc. **16** (1940), 195–200, 354–360, 442–448, 505–511, V, Proc. **18** (1942), 446–451.

Then, the difference $D\frac{dx^\lambda}{ds}$ between the two vectors $\frac{d\bar{x}^\lambda}{d\bar{s}}$ and $\frac{dx^\lambda}{ds} -\varepsilon\{{}^{\,\lambda}_{\mu\nu}\}\frac{dx^\mu}{ds}\xi^\nu$ which is obtained by a parallel displacement of $\frac{dx^\lambda}{ds}$ from the point $x^\lambda(s)$ to $\bar{x}^\lambda(\bar{s})$, is given by

$$D\frac{dx^\lambda}{ds}=\frac{dx^\lambda}{ds}\left(\frac{ds}{d\bar{s}}-1\right)+\varepsilon\left(\frac{d\xi^\lambda}{ds}\frac{ds}{d\bar{s}}+\{{}^{\,\lambda}_{\mu\nu}\}\xi^\mu\frac{dx^\nu}{ds}\right), \tag{2.3}$$

$\{{}^{\,\lambda}_{\mu\nu}\}$ being the Christoffel symbols of the second kind.

Calculating the $ds/d\bar{s}$ in (2.3), we ohtain, from (2.2),

$$\frac{ds}{d\bar{s}}=1-\varepsilon g_{\mu\nu}\frac{\delta\xi^\mu}{\delta s}\frac{dx^\nu}{ds} \tag{2.4}$$

to the first order, $\delta/\delta s$ denoting the covariant differentiation along the curve $x^\lambda(s)$. Thus, the equations (2.3) become

$$D\frac{dx^\lambda}{ds}=\varepsilon\left[\frac{\delta\xi^\lambda}{\delta s}-g_{\mu\nu}\frac{\delta\xi^\mu}{\delta s}\frac{dx^\nu}{ds}\frac{dx^\lambda}{ds}\right] \tag{2.5}$$

which shows that a necessary and sufficient condition that the infinitesimal deformation $\bar{x}^\lambda=x^\lambda+\varepsilon\xi^\lambda$ be a parallel tangent deformation for the curve $x^\lambda(s)$ is that

$$\frac{\delta\xi^\lambda}{\delta s}=\psi\frac{dx^\lambda}{ds}, \tag{2.6}$$

where ψ is necessarily equal to $g_{\mu\nu}\frac{\delta\xi^\mu}{\delta s}\frac{dx^\nu}{ds}$ because of the identity $g_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}=1.$

Let us now suppose that the vector ξ^λ is defined in all the space, that is, all the points x^λ in V_n are displaced to $\bar{x}^\lambda=x^\lambda+\varepsilon\xi^\lambda(x)$ by an infinitesimal deformation, and that every deformation defined by ξ^λ for any curve is always a parallel tangent deformation of this curve. Then from (2.6), we have $\xi^\lambda_{;\nu}\frac{dx^\nu}{ds}=\xi_{\mu;\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}\frac{dx^\lambda}{ds}$, from which $\frac{1}{2}(\xi_{\mu;\nu}+\xi_{\nu;\mu})=\frac{1}{n}\xi^\alpha_{;\alpha}g_{\mu\nu}$, thus, $\psi=\xi_{\mu;\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}=\frac{1}{n}\xi^\alpha_{;\alpha}$ being a point function, we have

$$\xi^\lambda_{;\nu}=\psi\delta^\lambda_\nu, \tag{2.7}$$

where the semi-colon denotes the covariant derivative with respect to $\{{}^{\,\lambda}_{\mu\nu}\}$. The equation (2.7) written in the covariant form

$$\xi_{\mu;\nu}=\psi g_{\mu\nu} \tag{2.8}$$

shows that the vector ξ_μ is a gradient of a function $F(x)$, say

$$\xi_\mu=\frac{\partial F}{\partial x^\mu}. \tag{2.9}$$

Putting $F=\frac{1}{\rho}$, we have from (2.8)

$$\rho_{\mu;\nu}-\rho_\mu\rho_\nu=\phi g_{\mu\nu}\,, \tag{2.10}$$

where $$\rho_\mu=\frac{\partial\log\rho}{\partial x^\mu}\quad\text{and}\quad\phi=-\frac{\psi}{F}.$$

The equations (2.10) show that our Riemannian space V_n admits a concircular transformation $\bar{g}_{\mu\nu}=\rho^2 g_{\mu\nu}$. One of the present authors has shown in a previous paper[1] that a necessary and sufficient condition that a V_n admits a concircular transformation is that V_n contains ∞^1 family of totally umbilical hypersurfaces whose orthogonal trajectories are geodesic Ricci-curves. It is easily shown that in such a case all the totally umbilical hypersurfaces are of constant mean curvature. For, from the Codazzi equations for the hypersurfaces $x^\lambda=x^\lambda(x^i)$ $(i,j,k,\dots=\dot{1},\dot{2},\dots,\dot{n}-\dot{1})$

$$H_{jk;h}-H_{jh;k}=B_\lambda B_j^{\cdot\mu}B_k^{\cdot\nu}B_h^{\cdot\omega}R^{\lambda}_{\cdot\mu\nu\omega}\,,$$

where H_{jk} denotes the second fundamental tensor, B_λ the unit normal, $R^{\lambda}_{\cdot\mu\nu\omega}$ the curvature tensor and $B_j^{\cdot\mu}=\frac{\partial x^\mu}{\partial x^j}$, we can deduce

$$g_{jk}H_{;h}-g_{jh}H_{;k}=B_\lambda B_j^{\cdot\mu}B_k^{\cdot\nu}B_h^{\cdot\omega}R^{\lambda}_{\cdot\mu\nu\omega}\,, \tag{2.11}$$

the second fundamental tensor H_{jk} having the form

$$H_{jk}=g_{jk}H\,.$$

Contracting g^{jk} to (2.11), we have

$$(n-2)H_{;h}=B_\lambda R^{\lambda}_{\cdot\omega}B_h^{\cdot\omega}\,, \tag{2.12}$$

because of the identities

$$g^{jk}B_j^{\cdot\mu}B_k^{\cdot\nu}=g^{\mu\nu}-B^\mu B^\nu\,,\quad B_\lambda B^\mu R^{\lambda}_{\cdot\mu\nu\omega}=0\quad\text{and}\quad g^{\mu\nu}R^{\lambda}_{\cdot\mu\nu\omega}=R^{\lambda}_{\cdot\omega}\,,$$

where $R^{\lambda}_{\cdot\omega}$ denotes the mixed Ricci-tensor, say, $R^{\lambda}_{\cdot\omega}=g^{\lambda\nu}R_{\nu\omega}$.

The equations (2.12) show that, in the case $n>2$, if the normal B_λ is in the Ricci-direction, then the mean curvature H is constant and vice versa.

Thus, we have the

Theorem: In order that a Riemannian space V_n $(n>2)$ admits an infinitesimal deformation which gives a parallel tangent deformation for any curve in V_n, it is necessary and sufficient that the V_n contains ∞^1 family of totally umbilical hypersurfaces of constant mean curvature whose orthogonal trajectories are geodesics.

The expression for the constant mean curvature may be easily found from the equation (2.8) as follows: The hypersurfaces $F=$ constants being totally umbilical and of constant mean curvature, we have, by covariant differentiation of this equation along the hypersurface $F=$const. or $x^\lambda=x^\lambda(x^i)$,

1) K. Yano: Concircular geometry II, loc. cit.

(2.13) $\xi_\mu B_j^{\cdot\mu}=0$ and $\xi_{\mu;\nu}B_j^{\cdot\mu}B_k^{\cdot\nu}+\xi_\mu g_{jk}HB^\mu=0$,

hence $(g^{\mu\nu}\xi_\mu\xi_\nu)_{;k}=2g^{\mu\nu}\xi_{\mu;\omega}\xi_\nu B_k^{\cdot\omega}=2\psi\xi_\omega B_k^{\cdot\omega}=0$,

and we know that the magnitude ξ of the ξ^λ is constant along the hypersurface F=const. The second equation of (2.13) and (2.8) give us

$$H=-\frac{\psi}{\xi}, \tag{2.14}$$

hence, H being constant, ψ is also constant along the hypersurfaces F=const.

3. One of the present authors has studied in a previous paper[1] the concurrent vector field in a Riemannian space. The necessary and sufficient condition that the vector field ξ^λ be a concurrent one is that

$$(\alpha\xi^\lambda)_{;\nu}+\delta_\nu^\lambda=0, \tag{3.1}$$

where α is a suitable scalar. Writing as ξ^λ instead of $\alpha\xi^\lambda$ we have

$$\xi^\lambda_{;\nu}=-\delta_\nu^\lambda. \tag{3.2}$$

In this case, the end point of the vector ξ^λ represents just the point fixed by the group of holonomy.

The condition (3.2) is quite analogous to (2.7), hence, if ξ^λ is a concurrent vector field satisfying (3.2), then the vector ξ^λ defines a parallel tangent deformation discussed in the Paragraph 2. But conversely, ξ^λ being a vector field defining a parallel tangent deformation, the vector ξ^λ gives not necessarily a concurrent vector field.

Let ξ^λ be a vector definining a parallel tangent deformation, that is to say, satisfying the equation (2.7). Then, all the hypersurfaces F=constants are totally umbilical and of constant mean curvature, and consequently by a well known theorem[2], the normals to the hypersurfaces F=constants are concurrent along the hypersurfaces, say, ξ^λ is concurrent along the hypersurfaces F=constants. In order that $\xi^\lambda(x)$ be a concurrent vector field in the space, it must be also concurrent in the direction normal to the hypersurfaces and its end point must coincide with the center of mean curvature passing through the point x^λ.

Thus, in order that the ξ^λ be a concurrent vector field in the space, it must be of the form

$$\xi^\lambda=\frac{1}{H}B^\lambda, \tag{3.3}$$

B^λ being the unit normal to the hypersurfaces and H the constant mean curvature. From (3.3), we have $\xi H=1$, then the equation (2.14) gives us $\psi=-1$. Thus, in order that a vector ξ^λ defining a parallel

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1) K. Yano: Sur le parallélisme et la concourance dans l'espace de Riemann, Proc. **19** (1943), 189-197.

2) K. Yano, loc. cit.

tangent deformation be a concurrent one, it is necessary and sufficient that the scalar ψ appearing in $\xi_{\mu;\nu}=\psi g_{\mu\nu}$ be constant at all the points of the space.

The curve generated by the vector ξ^λ being a geodesic, if ξ^λ denotes the point $\frac{1}{H}B^\lambda$ fixed by the group of holonomy, the magnitude ξ of ξ^λ must be of the form

$$\xi=c-s, \tag{3.4}$$

s being the arc length of the geodesic, and c an arbitrary constant. Thus, we have form (2.14) and $\psi=-1$

$$\frac{1}{H}=c-s,$$

which shows that the lengths of the geodesic orthogonal trajectories between the two of the hypersurfaces are constant and equal to the difference of the constant radius of mean curvature of these totally umbilical hypersurfaces. Conversely, if these conditions are satisfied, the vector field ξ^λ defined by (3.3) at all the points of the space is a concurrent vector field.

From the above considerations, we know that if ξ^λ is a concurrent vector field in V_n, then V_n must contain ∞^1 family of hypersurfaces totally umbilical and of constant mean curvature whose orthogonal trajectories are geodesics, and the lengths of geodesic orthogonal trajectories contained in two of these hypersurfaces are constant and equal to the difference of the constant radius of mean curvature of these hypersurfaces. Conversely, if these conditions are satisfied, the direction normal to these hypersurfaces is concurrent along these hypersurfaces and also in the direction normal to the hypersurfaces. Thus we have the

Theorem: In order that a Riemannian space V_n admits a concurrent vector field, it is necessary and sufficient that the V_n contains ∞^1 family of hypersurfaces totally umbilical and of constant mean curvature whose orthogonal trajectories are geodesics and the arc lengths of these geodesic orthogonal trajeetories contained in two of these hypersurfaces are constant and equal to the difference of the constant radius of mean curvature of the two hypersurfaces.

This theorem must replace the theorem 5.4 in the above cited paper which contains an error.