28. *A Kinematic Theory of Turbulence**.

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1. *Generalities.* In the theory of turbulence[1] the deviation of the velocity from its mean may be considered as a system of random vectors $u_\lambda(t, \mathfrak{X}, \omega)$, $\lambda=1,2,3$, where $t(\in R)$ is the time parameter and $\mathfrak{X}(\in R^3)$ denotes the position and $\omega\big(\in(\Omega, P)\big)$ is the elementary event. Then we have

$$(1) \qquad \mathscr{E}_\omega\big(u_\lambda(,\mathfrak{X},\omega)\big)=0\,.$$

When the system $\{u_\lambda(t,\mathfrak{X},\omega)\}$ is of Gaussian type[2], we say that the turbulence is of Gaussian type.

Now we define the moment tensor of the turbulence by

$$(2) \qquad R_{\lambda\mu}(t, \mathfrak{X}\,;\, s, \mathfrak{Y})=\mathscr{E}_\omega\{u_\lambda(t, \mathfrak{X}, \omega)u_\mu(s, \mathfrak{Y}, \omega)\}\,.$$

Then $R_{\lambda\mu}(t, \mathfrak{X}\,;\, s, \mathfrak{Y})$ is a positive-definite function of $(\lambda, t, \mathfrak{X})$ and $(\mu, s, \mathfrak{Y})$ in the sense of Bochner, namely we have

$$(3) \qquad R_{\lambda\mu}(t, \mathfrak{X}\,;\, s, \mathfrak{Y})=R_{\mu\lambda}(s, \mathfrak{Y}\,;\, t, \mathfrak{X}) \quad \text{and}$$

$$(4) \qquad \textstyle\sum_{ij}\xi_i\xi_j R_{\lambda_i\lambda_j}(t_i, \mathfrak{X}_i\,;\, t_j, \mathfrak{X}_j)\geqq 0\,;$$

in fact (3) is evident by (2) and the left side of (4) is equal to $\mathscr{E}_\omega\Big\{\big(\sum_i\xi_i u_{\lambda_i}(t_i, \mathfrak{X}_i, \omega)\big)^2\Big\}$. Conversely the function $R_{\lambda\mu}(t, \mathfrak{X}\,;\, s, \mathfrak{Y})$ satisfying (3) and (4) may be considered as the moment tensor of a turbulence of Gaussian type[3].

A turbulence is defined as temporally homogeneous, if its moment tensor satisfies

$$(5) \qquad R_{\lambda\mu}(t+\tau, \mathfrak{X}\,;\, s+\tau, \mathfrak{Y})=R_{\lambda\mu}(t, \mathfrak{X}\,;\, s, \mathfrak{Y})\,.$$

It is defined as spatially homogeneous, if we have

$$(6) \qquad R_{\lambda\mu}(t, \mathfrak{X}+\mathfrak{a}\,;\, s, \mathfrak{Y}+\mathfrak{a})=R_{\lambda\mu}(t, \mathfrak{X}\,;\, s, \mathfrak{Y})\,.$$

We say that it is isotopic if we have always

$$(7) \qquad \sum_{\lambda'\mu'} k_{\lambda'\lambda}k_{\mu'\mu}R_{\lambda'\mu'}\big(t, \mathfrak{X}\,;\, s, \mathfrak{X}+K(\mathfrak{Y}+\mathfrak{X})\big)=R_{\lambda\mu}(t, \mathfrak{X}\,;\, s, \mathfrak{Y})$$

for any orthogonal transformation $K\equiv\{k_{\lambda\mu}\,;\, \lambda, \mu=1,2,3\}$. We can easily prove by (3) that the isotropism implies the homogenuity.

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1) H. P. Robertson: The invariant theory of isotropic turbulence, Proc. Cambr. Phil. Soc. 36, 1940.

2) Cf. K. Itô: ガウス型確率變數系ニツイテ (全國紙上數學談話會第 261 號).

3) See Theorem 3 in my above-cited note (2).

It seems to be an important and perhaps difficult problem to determine the canonical form of $R_{\lambda\mu}(t, \mathfrak{X}; s, \mathfrak{Y})$ which satisfies (3), (4), (5) and (7).

2. *The temporally homogeneous and isotropic turbulence at a point.* For the investigation of this subject we can consider $u_\lambda(t, \omega)$ and $R_{\lambda\mu}(t, s)$ respectively instead of $u_\lambda(t, \mathfrak{X}, \omega)$ and $R_{\lambda\mu}(t, \mathfrak{X}; s, \mathfrak{Y})$. By (3) and (4) we have

$$(3') \quad R_{\lambda\mu}(t, s) = R_{\mu\lambda}(s, t) \qquad \text{and} \qquad (4') \quad \sum_{ij} \xi_i \xi_j R_{\lambda_i \lambda_j}(t_i, t_j) \geqq 0 \,.$$

The conditions (5) and (7) may be written in the forms:

$$(5') \quad R_{\lambda\mu}(t+\tau, s+\tau) = R_{\lambda\mu}(t, s) \quad \text{and} \quad (6') \quad \sum_{\lambda'\mu'} k_{\lambda'\lambda} k_{\mu'\mu} R_{\lambda'\mu'}(t, s) = R_{\lambda\mu}(t, s) \,.$$

Theorem 1. A necessary and sufficient condition that $R_{\lambda\mu}(t, s)$ should be the moment tensor of a temporally homogeneous and isotropic turbulence at a point is that $R_{\lambda\mu}(t, s)$ is expressible by the form:

$$(8) \qquad R_{\lambda\mu}(t, s) = \delta_{\lambda\mu} \int_{-0}^{\infty} \cos\big(\xi(t-s)\big) F(d\xi) \,,$$

where $\delta_{\lambda\mu}$ is the Kronecker's delta and F is a measure distribution on $[0, \infty)$ with the finite total measure.

Proof. Necessity. The isotropism (6′) implies that $R_{\lambda\mu}(t, s)$ is an invariant tensor. Therefore we obtain $R_{\lambda\mu}(t, s) = \delta_{\lambda\mu} C(t, s)$. From the temporal homogenuity (5′) and the symmetric character (3′) follows that $C(t, s)$ is a function of $|t-s|$ only, say $C_1(|s-t|)$. Now we see by (4′) that $C(|\tau|)$ is a positive-definite function of τ. Making use of the Bochner's theorem we obtain (8). The sufficiency is evident.

According to this theorem $u_\lambda(t, \omega)$ and $u_\mu(s, \omega)\, (\lambda \neq \mu)$ are non-correlated in this turbulence. Therefore, if we assume further that the turbulence be of Gaussian type, the three stochastic processès $\big(u_\lambda(t, \omega);\ -\infty < t < \infty\big)$, $\lambda = 1, 2, 3$, will become independent. Nevertheless each process is clearly a stationary process of Gaussian type with the correlation function $\rho(\tau) \equiv \int_{-0}^{\infty} \cos \tau\xi F(d\xi) / F\big([0 \infty)\big)$. In this case the problem may be reduced to the investigation of such a process.

Next we mention a theorem concerning the ergodicity of this process, which includes the result[1] obtained before by the author; the proof can be achieved by the same idea and so will be omitted.

Theorem 2. A necessarity and sufficient condition that a normalized continnous (in mean) stationary process $u(t, \omega)$ of Gaussian type should be ergodic in the strongly mixing type is that its correlation function $\rho(\tau)$ satisfies

$$(9) \qquad \lim_{\tau \to \infty} \rho(\tau) = 0 \,.$$

The condition (9) means that the correlation coefficient tends to 0

1) See K. Itô: On the ergodicity of a certain stationary process, Proc. **20** (1944), 54–55.

as the time interval increases indefinitely. In practice we may assume that it is well satisfied. Then by Theorem 2 we can see that

$$(10)\qquad P\left\{\omega\,;\ \lim_{T\to\infty}\frac{1}{T}\int_0^T u(t,\omega)u(t+\tau,\omega)\,dt=\rho(\tau)\right\}=1\,.$$

This identity justifies the practical method in which we make use of the time-mean of $u(t,\omega)u(t+\tau,\omega)$ at a certain (realized) value of ω instead of its mathematical expectation $\rho(\tau)$. It is also the case with turbulence.