41. *Normed Rings and Spectral Theorems, IV**.

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1. *Introduction.* The spectral theorem given in the third note[1] constitutes, in the special case when $T(M)$ is an enumerably-valued step function, a refinement with an error estimation of E. Schmidt's procedure of the approximate calculation of the greatest proper value of the integral equation. The result is then similar to that due to Temple and Collatz[2]. The purpose of the present note is to show that our treatment may be extended to obtain an approximate calculation of the lower proper values.

At this juncture, I intend to correct the misprints in III: 1) the right hand side of (iv) on page 72, line 5 must be read as

$$\frac{1}{\sqrt{\tau(\mu)}}\sqrt{\frac{F(T^{2n})}{F(T^{2(n+1)})}-\frac{F(T^{2n+1})^2}{F(T^{2(n+1)})^2}} \qquad (\text{as } n\to\infty)$$

2) $\lim_{n\to\infty}\tau_{n+1}(\mu-\varepsilon)=1$ on page 73, line 6 must be read as $\lim_{n\to\infty}\tau_{n+1}(\mu-\varepsilon)=\tau(\mu-0)=\tau(\mu)$.

2. *The Theorem.* As in III, let $\boldsymbol{R}$ be the totality of the real-valued continuous functions $S(M)$ on a bicompact Hausdorff space $\mathfrak{M}$, and let $F(S)$ be a positive linear functional on $\boldsymbol{R}$ such that $F(I)=1$ where $I(M)\equiv 1$. Then

$$F(T)=\int_{\mathfrak{M}}T(M)\varphi(dM)=\int_{\lambda_0}^{\lambda_1}\lambda d\tau(\lambda),\ \lambda_0=\inf_M T(M),\ \lambda_1=\sup_M T(M),$$
$$\tau(\lambda)=\varphi\big(M;\ T(M)<\lambda\big).$$

We assume that $\lambda_0>0$ and that $\tau(\lambda)$ be of the form

$$(\mathrm{A})\quad\begin{cases}\tau(\mu^{(2)}+0)>\tau(\mu^{(2)}-0) & \tau(\lambda)=\text{constant for } \mu^{(2)}<\lambda<\mu^{(1)}\\ \tau(\mu^{(1)}+0)>\tau(\mu^{(1)}-0), & \tau(\lambda)=\text{constant for } \mu^{(1)}<\lambda<\mu^{(0)}\\ \tau(\mu^{(0)}+0)>\tau(\mu^{(0)}-0), & \tau(\lambda)=\tau(\mu^{(0)}+0) \text{ for } \lambda>\mu^{(0)}\end{cases}$$

$\mu^{(0)}$, $\mu^{(1)}$ may respectively be called as *the maximal, the next maximal spectrum of F referring to T.*

We put

$$\frac{F(T^{2n+1})}{F(T^{2(n+1)})}=\frac{1}{\mu_n^{(0)}}\qquad \sqrt{\frac{F(T^{2n})}{F(T^{2(n+1)})}}=\frac{1}{\nu_n^{(0)}},$$

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* The cost of this research has been defrayed from the Scientific Research Expenditure of the Department of Education.

1) Proc. **20** (1944), 71. This note will be referred to as III.

2) G. Temple: Proc. London Math. Soc., **29** (1929), 257. L. Collatz: Math. Zeitschr., **46** (1940), 692. Our formula (iv) ((iv)′) does not contain the unknown value $\mu^{(1)}(\mu^{(2)})$ explicitely. Moreover in (iv)″ the values $\mu^{(0)}$, $\mu^{(1)}$, $\mu^{(2)}$ are only implicitely needed. These are the main difference of our results from Temple-Collatz's.

then from the proof of the spectral theorem in III we obtain

(i) $1/\nu_n^{(0)} \geqq 1/\nu_{n+1}^{(0)} \geqq 1/\mu^{(0)} \qquad (n \geqq 0)$,

(ii) $1/\nu_n^{(0)} \geqq 1/\mu_n^{(0)} \geqq 1/\mu^{(0)} \qquad (n \geqq 0)$,

(iii) $\lim_{n\to\infty} 1/\nu_n^{(0)} = \lim_{n\to\infty} 1/\mu_n^{(0)} = 1/\mu^{(0)}$,

(iv) $0 \leqq (1/\mu_n^{(0)}) - (1/\mu^{(0)}) \leqq \sqrt{(\nu_n^{(0)})^{-2} - (\mu_n^{(0)})^{-2}}$

if $$(1/\mu^{(1)}) - (1/\mu_n^{(0)}) > (1/\mu_n^{(0)}) - (1/\mu^{(0)}),$$

because, by (A),

$$\int \left(\frac{1}{\lambda} - \frac{1}{\mu_n^{(0)}}\right)^2 d\tau_{n+1}(\lambda) = \int \left(\frac{1}{\lambda} - \frac{1}{\mu_n^{(0)}}\right)^2 \frac{\lambda^{2(n+1)}}{F(T^{2(n+1)})} d\tau(\lambda) \geqq \left(\frac{1}{\mu^{(0)}} - \frac{1}{\mu_n^{(0)}}\right)^2$$

if $$(1/\mu^{(1)}) - (1/\mu_n^{(0)}) > (1/\mu_n^{(0)}) - (1/\mu^{(0)}).$$

We will prove the

Theorem. Put

$$F^{(1)}(S) = F\left(\frac{(\mu^{(0)}I - T)}{F(\mu^{(0)}I - T)} \cdot S\right),$$

and define

$$\frac{F^{(1)}(T^{2n+1})}{F^{(1)}(T^{2(n+1)})} = \frac{1}{\mu_n^{(1)}}, \qquad \sqrt{\frac{F^{(1)}(T^{2n})}{F^{(1)}(T^{2(n+1)})}} = \frac{1}{\nu_n^{(1)}},$$

then

(i)′ $1/\nu_n^{(1)} \geqq 1/\nu_{n+1}^{(1)} \geqq 1/\mu^{(1)} \qquad (n \geqq 0)$,

(ii)′ $1/\nu_n^{(1)} \geqq 1/\mu_n^{(1)} \geqq 1/\mu^{(1)} \qquad (n \geqq 0)$,

(iii)′ $\lim_{n\to\infty} 1/\nu_n^{(1)} = \lim_{n\to\infty} 1/\mu_n^{(1)} = 1/\mu^{(1)}$,

(iv)′ $0 \leqq (1/\mu_n^{(1)}) - (1/\mu^{(1)}) \leqq \sqrt{(\nu_n^{(1)})^{-2} - (\mu_n^{(1)})^{-2}}$

if $$(1/\mu^{(1)}) - (1/\mu_n^{(1)}) > (1/\mu_n^{(1)}) - (1/\mu^{(1)}).$$

Proof. $F^{(1)}(S)$ is a positive linear functional on R such that $F^{(1)}(I)=1$. Since, by the hypothesis concerning $\tau(\lambda)$,

$$F\left((\mu^{(0)}I - T)T^n\right) = \int_{\lambda_0}^{\lambda_1} (\mu^{(0)} - \lambda)\lambda^n d\tau(\lambda) = \int_{\lambda_0}^{\mu^{(1)}+0} (\mu^{(0)} - \lambda)\lambda^n d\tau(\lambda),$$

we have $F^{(1)}(T^n) = \int_{\lambda_0}^{\mu^{(1)}+0} \lambda^n d\tau^{(1)}(\lambda)$. Thus (i)′–(iv)′ may be proved as (i)–(iv).

3. *A practical formula.* (i)′–(iv)′ is applicable only when the exact value $\mu^{(0)}$ is obtained. We will give a practical approximate formula by making use of the approximations $\nu_k^{(0)}$, $\mu_k^{(0)}$, of $\mu^{(0)}$.

Lemma.

$$G(\nu) = \frac{F\left((\nu I - T)T^{2n+1}\right)}{F\left((\nu I - T)T^{2(n+1)}\right)} \quad \text{and} \quad H(\nu) = \frac{F\left((\nu I - T)T^{2n}\right)}{F\left((\nu I - T)T^{2(n+1)}\right)}$$

are decreasing in ν.

Proof. We have

$$G'(\nu)=\frac{F(T^{2n+2})^2-F(T^{2n+1})F(^{2n+3})}{F\left((\nu 1-T)T^{2(n+1)}\right)}\leqq 0\,,$$

since, by Schwartz's inequality,

$$F(T^{2n+2})=\int\lambda^{2n+2}d\tau(\lambda)\leqq\sqrt{\int\lambda^{2n+1}d\tau(\lambda)\int\lambda^{2n+3}d\tau(\lambda)}$$
$$=\sqrt{F(T^{2n+1})F(T^{2n+3})}\,.$$

That $H'(\nu)\leqq 0$ will be proved similarly. Q. E. D.

By (i)–(iv) we have

$$\mu_k^{(0)}\leqq\mu^{(0)}\qquad(k\geqq 0)$$

$$\bar{\mu}_k^{(0)}\geqq\mu^{(0)}\qquad\left(\text{if }(1/\mu^{(1)})-(1/\mu_k^{(0)})>(1/\mu_k^{(0)})-(1/\mu^{(0)})\right),$$

where $$1/\bar{\mu}_k^{(0)}=(1/\mu_k^{(0)})-\sqrt{(\nu_k^{(0)})^{-2}-(\mu_k^{(0)})^{-2}}$$

Hence, by the lemma and (i)′–(iv)′, we have a *Practical formula*:

$$0\leqq\frac{F\left((\mu_k^{(0)}I-T)T^{2n+1}\right)}{F\left((\mu_k^{(0)}I-T)T^{2(n+1)}\right)}-\frac{1}{\mu^{(1)}}\qquad\text{which is}$$

$$\leqq\sqrt{\frac{F\left((\mu_k^{(0)}I-T)T^{2n}\right)}{F\left((\mu_k^{(0)}I-T)T^{2(n+1)}\right)}-\frac{F\left((\bar{\mu}_k^{(0)}I-T)T^{2n+1}\right)^2}{F\left((\bar{\mu}_k^{(0)}I-T)T^{2(n+1)}\right)^2}}$$
$$+\frac{F\left((\mu_k^{(0)}I-T)T^{2n+1}\right)}{F\left((\mu_k^{(0)}I-T)T^{2(n+1)}\right)}-\frac{F\left((\bar{\mu}_k^{(0)}I-T)T^{2n+1}\right)}{F\left((\bar{\mu}_k^{(0)}I-T)T^{2(n+1)}\right)}$$

for sufficiently large k,n, viz. if k,n are large such that

$$\text{(C)}\quad\begin{cases}(1/\mu^{(1)})-(1/\mu_k^{(0)})>(1/\mu_k^{(0)})-(1/\mu^{(0)})\,,\\(1/\mu^{(2)})-(1/\mu_n^{(1)})>(1/\mu_n^{(1)})-(1/\mu^{(1)})\,.\end{cases}$$

Remark. It is to be noted that, in practice, the condition (C) is often satisfied only when $k,n\geqq 3$ or 4. Our procedure may be repeated to obtain the approximations to the next smaller proper values $\mu^{(2)}$ etc.